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PHYSICAL SOCIETY OF LONDON.

From December 1919 *to August* 1920.

VOL. XXXII.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1919-1920.

October 24, 1919.

Meeting held at Imperial College of Science.

Prof. C. H. LEES, F.R.S., President, in the Chair.

The following Papers were read :—

1. "The Effect of Pressure and Temperature on a Meter for Measuring the Rate of Flow of a Gas." By Dr. N. W. McLACHLAN.

2. "A Cheap and Simple Micro-Balance." By Capt. J. H. SHAXBY.

3. "The Resolution of a Curve into a Number of Exponential Components." By J. W. T. WALSH.

November 14, 1919.

Meeting held at the Imperial College of Science.

Prof. C. H. LEES, F.R.S., President, in the Chair.

The following Papers were read :—

1. "On the Self-inductance of Single-layer Flat Coils." By S. BUTTERWORTH.

2. "An Experimental Method of Determining the Primary Current at Break in a Magneto." By Dr. N. W. McLachlan.

A New Form of Wehnelt Interrupter was shown by F. H. Newman.

November 28, 1919.

Meeting held at Imperial College of Science.

Prof. C. H. Lees, F.R.S., President, in the Chair.

A Discussion on Lubrication was held.

December 12, 1920.

Meeting held at Imperial College of Science.

Prof. C. H. Lees, F.R.S., President, in the Chair.

The following Papers were read :—

1.*" First Steps in the Experimental Analysis of a Galvanic Cell." By Prof. W. M. Coleman.

2. " Radiation from a Perfectly Diffusing Circular Disc." By J. W. T. Walsh.

3. " Testing of Thermionic Valves for Non-passage of Reverse Current at High Voltage." By Dr. N. W. McLachlan.

4. " On Recording and Reproducing Sounds by Means of Light." By Prof. A. O. Rankine.

* Read by Dr. D. Owen, in the absence of the Author.

January 7 and 8, 1920.

The Tenth Annual Exhibition of Scientific Apparatus was held in conjunction with the Optical Society.

Short Lectures were given on both days by

Prof. F. J. CHESHIRE, on "Some Polarisation Experiments," and

Prof. A. O. RANKINE, on "The Use of Light in the Transmission and Reproduction of Speech."

January 23, 1920.

Meeting held at City and Guilds Technical College, Finsbury.

Prof. C. H. LEES, F.R.S., President, in the Chair.

The following Papers were read :—

1. "On Some Experiments in which Two Neighbouring Maintained Oscillatory Circuits Affect a Resonating Circuit." By Dr. J. H. VINCENT.

2. "Measurement of the Chief Parameters of Triode Valves." By Prof. W. H. ECCLES.

3. "Measurement of the Amplification of a Radio-Frequency Amplifier." By Mr. F. W. JORDAN.

4. "Measurement of the Amplification Given by Triode Amplifiers at Audible and Radio Frequencies." By F. E. SMITH and the late H. C. NAPIER.

The following were exhibited :—

(1) An Instrument for Indicating the Hardening Temperature of Steel. By the Hon. C. W. STOPFORD and C. R. DARLING.

(2) A Thermo-electric Cell of Constant Voltage. By C. R. DARLING.

February 13, 1920.

Annual General Meeting.

Held at Imperial College of Science.

Prof. C. H. **Lees** delivered the Presidential Address, taking as his subject " The Temperature of the Earth's Interior."

A Paper on " The Influence of Small Changes of Temperature on Atmospheric Refraction " was read by Sir **Arthur Schuster**.

GENERAL BUSINESS.

It was moved by the **President** and seconded by Dr. **Borns** that the reports of the Secretary and Treasurer be adopted. This was unanimously carried.

DRAFT REPORT OF COUNCIL.

In the year 1919, 13 ordinary Science Meetings and one Special General Meeting of the Society have been held. Of these the Special General Meeting and 12 ordinary Meetings were held at the Imperial College of Science, and one ordinary Meeting, that on July 11, was held at the National Physical Laboratory by invitation of Sir Richard Glazebrook, K.C.B., F.R.S. The average attendance at the Meetings was 53, as compared with 35 during the preceding year.

At a Special General Meeting held on June 13, certain proposals made by the Institute of Physics were recommended for adoption by the Physical Society as one of the participating Societies. These proposals which affect subscriptions and the cost of certain publications were adopted unanimously and will become operative when the Institute of Physics is fully established.

The ordinary Meeting on March 28 was devoted to a discussion on Metrology and its application to the Industries. It was very successful and was attended by 91 Fellows and Visitors.

On November 28 a discussion on Lubrication was held. There were numerous contributions to the discussion, and more than 100 Fellows and Visitors were in attendance.

There have been two resignations of Officers of the Society. Dr. Allen has been appointed to Edinburgh University and resigned the Secretaryship in June. Prof. S. W. J. Smith, a former Secretary and for many years Librarian of the Society, resigned the latter position on his appointment as Professor of Physics at Birmingham University. He is succeeded by Dr Rankine as Librarian.

During the Session the Society arranged for the publication of a second edition of Prof. Eddington's Report on the " Relativity Theory of Gravitation," and for a Report by Prof. Fowler, F.R.A.S., F.R.S., on " Series in Spectra "

The number of Honorary Fellows on the roll on December 31, 1919, was 10, and the number of ordinary Fellows was 484, an increase of 39 on the number of last year. Fifty new Fellows and 7 Students have been elected, and 3 Fellows have resigned.

The Society has to mourn the loss by death of 8 Fellows, namely, Prof. Carey Foster, Prof. W. Watson, Sir William Crookes, Lord Rayleigh, Mr. D. Rintoul, Mr. F. J. Fry, Mr. W. R. Phelips, and S. D. Chalmers. Prof. Carey Foster was a past President and was one of the founders of the Society. Prof. W. Watson was for many years a Secretary of the Society and afterwards served as Librarian.

REPORT OF THE TREASURER.

The accounts for the past year show a great improvement in the financial position of the Society. The total income is £1,261. 2s. 7d., as against £1,055. 8s. 0d. for the previous year. This is due partly to the growth in membership leading to increased subscriptions, but in the main to the larger sale of publications. The amount realised by sales was £280. 16s. 10d., as against £148. 2s. 4d. in 1918.

On the other hand, more matter has been published in the "Proceedings," and, although no special publications have been issued during the year, the amount spent on printing has increased. Other items have not varied seriously, and thus there is a substantial balance of £273. 15s. 11d. of income over expenditure.

In the annual accounts hitherto it has been customary to include the cost of the publications over a complete volume (or session), which does not coincide with the financial year. The Council have now decided that it is preferable in all cases to include in the accounts the expenditure which has actually been incurred during the particular year, as distinct from the session, so that the expenditure should truly correspond with the revenue and with the other items of expenditure. Consequently there is a sum of £95. 6s. 0d., mainly expended on Part I. of Vol. XXXI. of the "Proceedings" (at the end of 1918), which does not appear in the accounts of either 1918 or 1919.

The Balance Sheet shows that the position in regard to arrears of subscriptions is increasingly unsatisfactory. The sum recovered during the past year has been much less than was hoped, and for this session a heavy reserve has been set against the sum due.

The investments, as in previous years, have been valued at market prices through the courtesy of the London, County, Westminster & Parr's Bank. They show a further depreciation in value. During the year under review the Exchequer Bonds and National War Bonds were converted into Funding Loan, and a further sum of £20 was invested in the latter to bring the total up to a round figure. The cash on deposit at the Bank has increased by £200.

The accounts as a whole show that the normal activities of the Society are not likely to suffer in the future through lack of income.

INCOME AND EXPENDITURE ACCOUNT.

From January 1st to December 31st, 1919.

Income.	£	s.	d.	£	s.	d.
Entrance Fees	49	7	0			
Subscriptions by Fellows	537	12	0			
,, Voluntary	16	16	0			
,, by Students	2	12	6			
,, Arrears paid	33	14	6			
,, Paid in Advance	41	17	6			
,, for "Science Abstracts" and Advance Proofs	19	5	0			
				701	4	6
Composition Fees				94	10	0
Dividends :—						
Furness Debenture Stock	11	3	10			
Midland Railway	28	5	0			
Metropolitan Board of Works	7	0	0			
Lancaster Corporation Stock	8	8	0			
New South Wales Stock	8	17	10			
London, Brighton & South Coast Railway	18	12	6			
Great Eastern Railway	14	0	0			
India 3½% Stock	17	10	0			
Exchequer Bonds, 5%	14	4	4			
National War Bonds, 5%	3	12	2			
Funding Loan	6	12	9			
				138	6	5
Income Tax refunded				38	0	1
Interest on deposit account				8	4	9
Sales of Publications (Fleetway Press, Ltd.)				280	16	10
				£1,261	2	7

Expenditure.	£	s.	d.	£	s.	d.
"Science Abstracts"	282	12	0			
,, ,, Extra copies	4	4	0			
				286	16	0
Fleetway Press, Ltd. :—						
"Proceedings"	428	2	11			
Bulletin	50	18	8			
Distribution (Postage)	50	14	5			
General	52	0	6			
				581	16	6
Periodicals	1	15	0			
Reporting	47	5	8			
Refreshments and Attendance	21	10	9			
Petty Cash—						
Secretaries' Expenses	14	10	11			
Treasurer's Expenses	4	11	4			
Insurance	3	2	6			
Advertising	3	14	0			
Royal Asiatic Society	2	2	0			
Bank Charges		2	0			
				98	14	2
Donations—						
Conjoint Board of Scientific Societies	10	0	0			
Annual Tables of Constants	10	0	0			
				20	0	0
Balance, being excess of income over expenditure				273	15	11
				£1,261	2	7

W. R. COOPER, *Honorary Treasurer.*

Audited and found correct,

W. A. J. O'MEARA,
G. W. O. HOWE, } *Honorary Auditors.*

February 5th, 1920.

BALANCE SHEET AT DECEMBER 31ST, 1919.

ASSETS.	£	s.	d.	£	s.	d.
Subscriptions in arrears	255	3	0			
Less reserve for subscriptions probably unrealisable	150	0	0			
				105	3	0
Investments (valued at Dec. 31):—						
£533 Furness 3 per cent. Debenture Stock	261	0	0			
£1,600 Midland Railway 2½ per cent. Perpetual Preference Stock	680	0	0			
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	162	0	0			
£400 Lancaster Corporation 3 per cent. Redeemable Stock	224	0	0			
£254 2s. 9d. New South Wales 3½ per cent. Ordinary Stock	218	0	0			
£500 London, Brighton & South Coast Railway Ordinary Stock...	350	0	0			
£500 Great Eastern Railway 4 per cent. Debenture Stock	335	0	0			
£500 India 3½ per cent. Stock......	305	0	0			
£650 4% Funding Loan, 1960-90	494	0	0			
				3,029	0	0
Stock of Publications (Treasurer's valuation)				250	0	0
Cash at Bank on Deposit				250	0	0
Cash at Bank, Current Account at Dec. 31	218	3	11			
Adjustment for outstanding cheques	121	5	0			
				96	18	11
Cash in hand (Treasurer's Petty Cash)				1	5	5
				£3,732	7	4

LIABILITIES.	£	s.	d.
Life Compositions	1,946	10	0
Balance, General Fund	1,785	17	4
	£3,732	7	4

W. R. COOPER, *Honorary Treasurer.*

Audited and found correct,

W. A. J. O'MEARA, } *Honorary Auditors.*
G. W. O. HOWE, }

February 5th, 1920.

LIFE COMPOSITION FUND AT DECEMBER 31ST, 1919.

	£	s.	d.
145 Fellows paid £10	1,450	0	0
3 Fellows paid £15	45	0	0
8 Fellows paid £21	168	0	0
9 Fellows paid £31. 10s.	283	10	0
	£1,946	10	0

Audited and found correct,

W. R. COOPER, *Honorary Treasurer*.

W. A. J. O'MEARA,
G. W. O. HOWE, } *Honorary Auditors.*

February 5th, 1920.

On behalf of the Council, the President moved: "That this General Meeting of the Physical Society of London approves of the foundation of an International Union of Physics."

This was seconded by Mr. Guild and carried unanimously.

The election of Officers and Council for the ensuing year resulted as follows:—

President.—Sir W. H. Bragg, C.B.E., M.A., F.R.S.

Vice-Presidents, who have filled the office of President.—Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, C.B., M.A., F.R.S.; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S.; Sir Oliver J. Lodge, D.Sc., F.R.S.; Sir Richard Glazebrook, K.C.B., D.Sc., F.R.S.; Prof. J. Perry, D.Sc., F.R.S.; C. Chree, Sc.D., LL.D., F.R.S.; Prof. H. L. Callendar, M.A., LL.D., F.R.S.; Sir Arthur Schuster, Ph.D., Scd., F.R.S.; Sir J. J. Thomson, O.M., D.Sc., F.R.S.; Prof. C. Vernon Boys, F.R.S.; Prof. C. H. Lees, D.Sc., F.R.S.

Vice-Presidents.—H. S. Allen, M.A., D.Sc.; Prof. W. Eccles, D.Sc.; Prof. A. S. Eddington, M.A., M.Sc., F.R.S.; R. S. Willows, M.A., D.Sc.

Secretaries.—D. Owen, B.A., D.Sc.; F. E. Smith, O.B.E., F.R.S.

Foreign Secretary.—Sir Arthur Schuster, Ph.D., ScD., F.R.S.

Treasurer.—W. R. Cooper, M.A., B.Sc.

Librarian.—A. O. Rankine, D.Sc.

Other Members of Council.—C. R. Darling, F.C.; PrI. C. L. Fortescue, O.B.E.; E. Griffiths, D.Sc.; E. H. Rayner, M.A., D.Sc.; A. Russell, M.A., D.Sc.; Prof. Sir Ernest Rutherford, D.Sc., F.R.S.; G. F. C. Searle, M.A., D.Sc., F.R.S.; T. Smith, B.A.; J. H. Vincent, D.Sc., M.A.; F. J. W. Whipple, M.A.

The customary votes of thanks concluded the business of the meeting.

February 27, 1920.

Meeting held at Imperial College of Science.

Prof. W. H. Bragg, F.R.S., President, in the Chair.

The following Papers were read :—

1. " On Balancing Errors of Different Orders." By T. Smith.

2. " Notes on the Testing of Bars of Magnet Steel." By Dr N. W. McLachlan.

3. " On the Forces Acting on Heated Metal Foil." By G. D. West.

Miss N. Hosali exhibited a number of Crystal Models.

March 12, 1920.

Meeting held at Imperial College of Science.

Prof. W. H. Bragg, F.R.S., President, in the Chair.

The following Papers were read :—

1. " The Absorption of Gases in a Discharge Tube." By F. W. Newman.

2. " A Directional Hot-wire Anemometer of High Sensitivity, especially Applicable to the Investigation of Slow Rates of Flow of Gases." By J. S. G. Thomas.

3. " Experiments with a New Micro-Balance." By Dr. Hans Pettersson.

March 26, 1920.

Meeting held at the Imperial College of Science.

Prof. W. H. Bragg, F.R.S., President, in the Chair.

A Discussion on Einstein's Theory of Relativity was held.

April 23, 1920.

Meeting held at Imperial College of Science.

Prof. W. H. **Bragg**, F.R.S., President, in the Chair.

The Fifth Guthrie Lecture was delivered by **Charles Edward Guillaume**; *subject*—

"The Anomaly of the Nickel Iron Alloys; its Causes and Applications."

May 14, 1920.

Meeting held at Imperial College of Science.

Prof. W. H. **Bragg**, F.R.S., President, in the Chair.

A Demonstration of Experiments on the Thermionic Properties of Hot Filaments was given by Dr. F. **Ll. Hopwood**.

The following Papers were read :—

1. "A Modified Theory of the Crookes Radiometer." By G. D. **West**.

2.* "On the Magnetic Properties of Silicon Iron (Stalloy) in Alternating Magnetic Fields of Low Value." By A. **Campbell**.

3. "On Tracing Rays through an Optical System." By T. **Smith**.

* Read by Mr. D. W. **Dye**, in the absence of the Author.

May 28, 1920.

Meeting held at Imperial College of Science.

Prof. W. H. **Bragg**, F.R.S., President, in the Chair.

A Discussion on X-Ray spectra was held.

June 11, 1920.

Meeting held at Imperial College of Science.

Prof. W. H. **Bragg**, F.R.S., President, in the Chair.

The following Papers were read :—

1. "Radiation and Convection from Heated Surfaces." By Dr. T. **Barratt** and Mr. A. J. **Scott**.

2. "An Electrical Hot-wire Inclinometer." By J. S. G. **Thomas**.

3. "Convective Cooling and the Theory of Dimensions." By L. F. **Richardson**.

4.*"The Radiation from a Perfectly Diffusing Circular Disc." By J. W. T. **Walsh**.

* Taken as read in the absence of the Author.

June 25, 1920.

Meeting held at Imperial College of Science.

Prof. W. H. **Bragg**, F.R.S., President, in the Chair.

The following Papers were read :—

1. "The Origin of the Elements." By Dr. J. H. **Vincent**.

2. "The Construction of Thermo-couples by Electro-deposition." By W. **Hamilton Wilson** and Miss T. D. **Epps**.

3. "The Use of Vacuum Arcs for Interferometry." By J. **Guild**.

4. *"The Maintenance of a Vibrating System by Means of a Triode Valve." By S. **Butterworth**.

* Read, in the absence of the Author, by Mr. R. L. **Smith-Rose**.

I. *The Effect of Pressure and Temperature on a Meter for Measuring the Rate of Flow of a Gas.* By N. W. McLachlan, *D.Sc., M.I.E.E.*

COMMUNICATED BY E. H. RAYNER, M.A., SC.D.
RECEIVED JUNE 19, 1919.

A TYPE of meter for measuring the rate of flow of a gas in litres per minute, or in any other unit of like dimensions, is shown diagrammatically in Fig. 1. It consists of a flat movable vane *V* (balanced and damped) on pivots, controlled by a spring *S*, the moving system being somewhat similar to that of a moving coil ammeter. The vane is deflected by the gas which issues from a nozzle *N*, and leaves the instrument case *C* by the exit tube *E*. The upper side of the case, the overall diameter being about 5·5 cm., is covered with glass, and the construction of the instrument is arranged so that there are no leaks when it is in use.

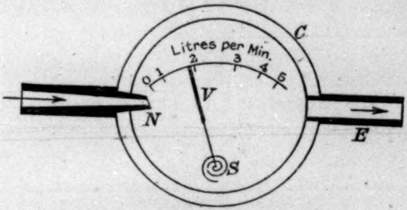

FIG. 1.—DIAGRAMMATIC SKETCH OF METER.

1. *Theory of Instrument.*

The force normal to the movable vane of the meter will be assumed as

$$F = k\rho A v^n, \qquad (1)$$

where k=constant,
ρ=density of gas,
A=effective area of nozzle,
v=velocity of gas jet, and
n=an index, which is shown hereafter to be constant.

The value of k depends on the area of the vane, the angle it makes with the gas jet, and the influence of the instrument case. In the experimental work described herein, the lines of flow of the gas were parallel, and the jet did not break up before it impinged on the vane.

Case (*a*).—When the "mass of gas" (equivalent to the volume at normal temperature and pressure) passing through the meter per minute is constant for all exit pressures, and the vane is fixed in position, we have the force normal to the vane,

$$\begin{aligned} F &= k(\rho A v)v^{n-1} \\ &= k \times \text{mass per min.} \times v^{n-1}. \quad . \quad . \quad . \qquad (2) \end{aligned}$$

Also

$$\begin{aligned} \rho &= \frac{\rho_0}{p_0}\, p\, \frac{273}{T}, \\ &= k_1 \frac{p}{T}, \end{aligned}$$

where ρ_0=density of gas at N.T.P.,
p_0=760 mm. Hg.,
p=exit pressure of gas,
T=abs. tempr. of gas °C.,

and

$$k_1 = \rho_0 \frac{273}{p_0}.$$

Since $\rho A v$ is constant, and A is assumed constant, it follows that

$$\rho v = \text{const.} = k_2,$$

or,

$$v = \frac{k_2}{\rho} = k_3 \frac{T}{p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where

$$k_3 = k_2 / k_1 = \text{const.}$$

From (3) it is clear that, provided the temperature is constant , the velocity of the gas increases with decrease in the pressure of the external atmosphere into which the gas is discharged. This pressure is taken to be equal to the static pressure of the gas jet. Since a decrease in the external pressure is equivalent to an increase in height in the isothermal atmosphere *—*i.e.*, an increase in altitude—the velocity of the gas jet increases with the altitude.

Substituting in (2) for v, we obtain

$$\begin{aligned} F &= k \times \text{mass} \times \left(k_3 \frac{T}{p}\right)^{n-1} \\ &= c\left(\frac{T}{p}\right)^{n-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4) \end{aligned}$$

*** In the work described below, the height of the isothermal atmosphere is taken. The relation between isothermal height and pressure is given by $h = 27{,}200 \log_e 760/p$ mm., where h is in feet. Ground level is taken as 760 mm. mercury and 10°C. The curve of Fig. 2 shows the relation graphically.**

where c=a constant, when the vane occupies a definite position with respect to the nozzle.

From (4) it follows that, provided the vane is fixed with reference to the nozzle, and the temperature is constant, the force normal to the vane increases with decrease in pressure—*i.e.*, increase in altitude, although the mass of gas flowing through the meter per minute is constant. Hence, if the vane were free to move, it would be deflected beyond its position at ground level. With a given pressure, corresponding to a definite altitude, a decrease in temperature, which is experienced in the upper atmosphere, is accompanied by a decrease in the force on the vane. This is due to an increase in density and a consequent reduction in velocity. In this case the deflection of the vane will diminish. Thus, in the upper atmosphere where a reduction in both temperature and pressure is encountered, the effect of the former will partially compensate that of the latter.

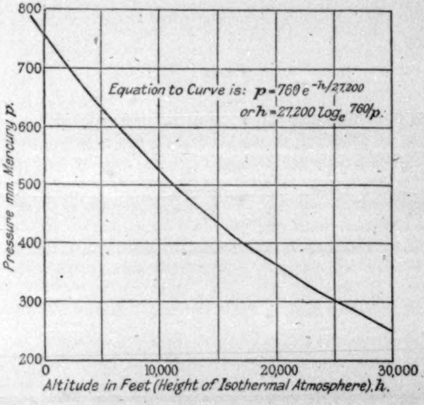

FIG. 2.—CURVE SHOWING RELATION BETWEEN HEIGHT OF THE ISOTHERMAL ATMOSPHERE AND THE PRESSURE.

Case (*b*).—When the meter reading is constant at all altitudes and temperatures, we have the force normal to the vane

$$F = k\rho A v^n = \text{a constant.} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting for ρ, $F=kk_1p/TAv^n$

$\therefore$ $v^n=c_1T/p$,

where c_1=a constant.

$\therefore$ $v^{n-1}=c_2(T/p)^{n'}$

where c_2=a constant,

and $n'=(n-1)/n$.

Thus, $(\rho Av)v^{n-1}=\text{mass}\times c_2\left(\frac{T}{p}\right)^{n'}=$ a constant; or mass $\propto (p/T)^{n'}$.

Since the mass of gas is proportional to the volume at N.T.P. we get

$$V=c_3\left(\frac{p}{T}\right)^{n'}, \qquad \ldots\ldots\ldots \quad (6)$$

where V=volume at N.T.P., and c_3=a constant.

If n' is constant at all pressures, for a given flow-meter reading and constant temperature, it will be seen from (6) that the mass of gas decreases with the pressure, and therefore with increase in altitude. When the pressure is constant the mass increases with decrease in temperature. Moreover, as stated previously, in connection with case (*a*), the effect of temperature will partially compensate that of altitude.

2. *Experiments at Low Pressures and Constant Temperature.*

In order to test the validity of the above theory, the experimental work described below was carried out. Since the scale of the meter from 0 to 5 litres per minute is only about 4 cm. long, the accuracy of the readings was not all that could be desired. Further, the vane of the meter oscillated when the flow exceeded 5 litres per minute, so that little or no reliance could be placed on observations above this figure.

The apparatus used during the experiments is shown diagrammatically in Fig. 3. An oxygen cylinder with reducing valve is connected through a throttle valve to the flow-meter, which is situated in an air-tight compartment or chamber fitted with glass windows. The gas from the flow meter is passed through a wet meter, and discharged thence into the chamber. A vacuum pump is used to keep the pressure of the gas in the chamber equal to that corresponding to some definite altitude. The pressure of the gas in the chamber is measured by means of a mercury manometer. In conducting a test at any particular altitude, the oxygen supply is adjusted so that the flow-meter reads a certain number of litres per minute—*i.e.*, the pointer is at a fixed mark on the scale. The control valve to the pump and the leak to the outer atmosphere are arranged to keep the pressure in the chamber constant. The time for two or more revolutions of the wet meter is taken. Knowing the time per revolution, the vapour pressure of the water in the meter, also the temperature and pressure of the gas in the chamber, the readings of the wet meter are reduced to normal temperature and pressure—*i.e.*, 0°C. and 760 mm. Hg. The calibration of the flow-meter obtained in this way is correct only when the gas used is oxygen at the same temperature and pressure as that of the calibration test. If the meter is used to measure the rate of flow of air, or any other gas, it will have to be recalibrated for the gas in question, or a correction made for the different density thereof, as shown in the appendix.

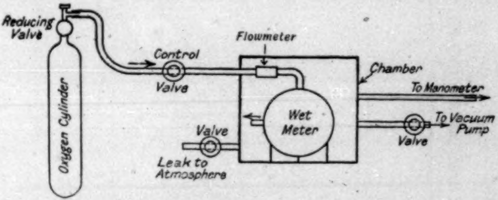

Fig. 3.—Diagram of Apparatus used for Experiments at Low Pressures.

3. *Results of Experiments at Low Pressures and Constant Temperature.*

The results of the experiments are illustrated graphically in Figs. 4, 5, 6 and 7. The calibrations of the meter at ground level (in this case −670 ft. = −200 metres on the isothermal atmosphere scale) and at various altitudes up to 30,000 ft. = 9,000 metres, are given. It appears from the results, that the ation between the N.T.P. volume of gas per minute and the ow-meter reading is very nearly linear at any altitude.

The relation between N.T.P. litres per minute and the altitude in feet for various flow-meter readings is shown in Fig. 6. The set of points for any particular flow-meter reading do not lie on a straight line, but on a curve of the form $y=ae^{-bx}$.

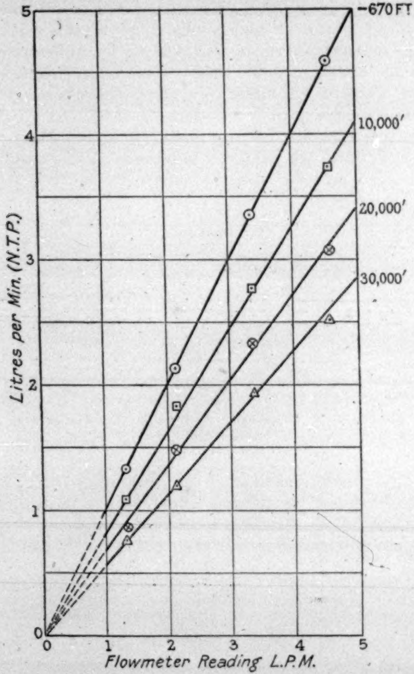

FIG. 4.—CALIBRATION OF METER AT VARIOUS ALTITUDES (EXPERIMENTAL).

The reason for this is outlined in the appendix. It may be sufficiently accurate, for practical purposes (since the accuracy of the meter itself is not great owing to its size), to draw the best straight lines through the sets of points. The slopes of the lines thus obtained diminish with the meter reading. This is due to a diminution in the value of c_3 in equation (6). When the above procedure is adopted, we may write

$$V+mh=V_0, \qquad (7)$$

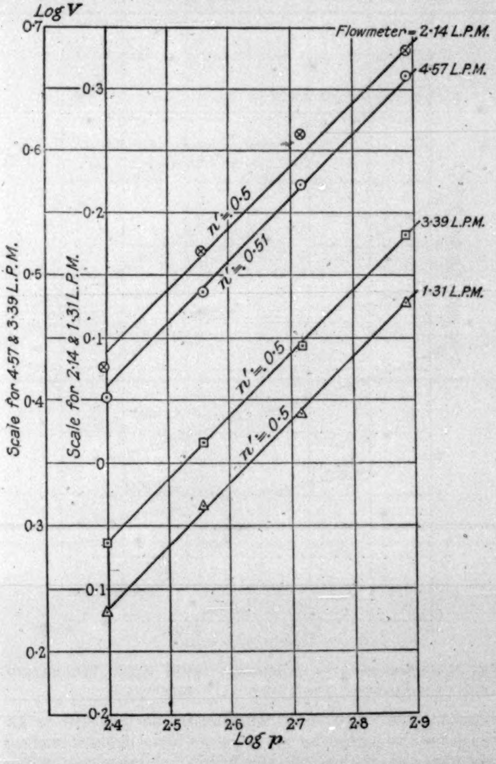

FIG. 5.—LOGARITHMIC PLOTTING OF N.T.P. VOLUME IN LITRES PER MINUTE, AND PRESSURE MM. OF HG. FOR VARIOUS METER READINGS.

where V_0=N.T.P. volume at ground level,
h=altitude in feet,
m=slope of line,
and V=N.T.P. volume at altitude h.

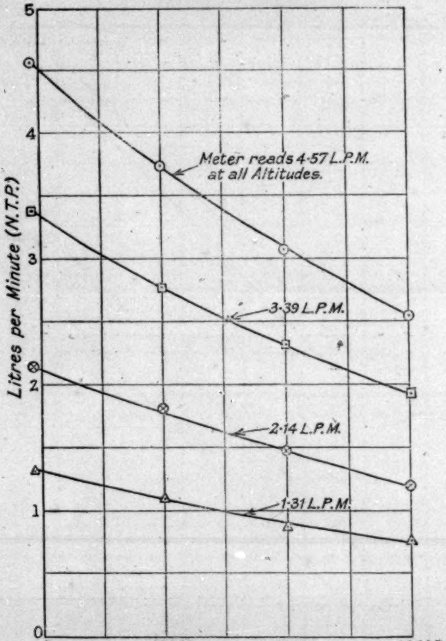

FIG. 6.—CURVES SHOWING RELATION BETWEEN N.T.P. VOLUME AND ALTITUDE FOR VARIOUS METER READINGS (EXPERIMENTAL).

Knowing the value of m, which must be regarded as an average for the particular set of points for a definite reading of the meter at all altitudes, the N.T.P. volume of gas at *any* altitude can be calculated.

In Fig. 7 the results of Fig. 6 are plotted on a pressure base, which causes an alteration in the curvature from convex upwards to concave upwards.

Log V is plotted against log p in Fig. 5. The sets of points for 1·31 and 3·39 litres per minute are almost collinear, but those for the other flow-meter readings lie on curves having a slight concavity upwards. This may be due to experimental errors. Taking the points to lie on straight lines in all cases, the slopes of these lines represent the values of n' in equation (6).

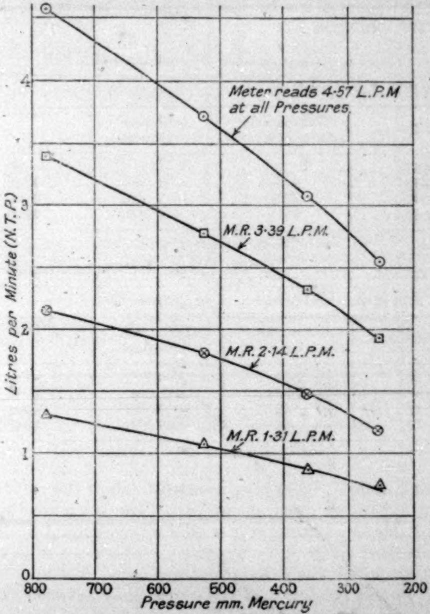

FIG. 7.—CURVES OF FIG. 6 ON A PRESSURE BASE.

Using the results obtained from Fig. 5, the value of n' may be taken as 0·5 on the average. The corresponding value of

n in equation (1) is 2. Thus the force on the vane, when the latter occupies a definite position with reference to the nozzle, varies approximately as the square of the velocity of the gas impinging on it. The slight variations in n' may be caused by the influence of the case, and also by interference effects due to the position of the vane with reference to the nozzle and to the exit pipe.

TABLE I.—*Meter Reading Throughout*=4·57 *L.P.M.*

N.T.P. litres per minute.		Altitude in feet (isothermal atmosphere).	Metres.
Experiment.	Calculation, assuming $n'=0{\cdot}5$		
4·57	4·57	−670	−200
3·73	3·76	+10,000	3,000
3·07	3·12	+20,000	6,000
2·53	2·6	+30,000	9,000

$T=11$°C, barometer=779 mm. Hg.

TABLE II.—*Meter Reading Throughout*=2·14 *Litres per Minute.*

N.T.P. litres per minute.		Altitude in feet (isothermal atmosphere).	Metres.
Experiment.	Calculation, assuming $n'=0{\cdot}5$		
2·14	2·14	−670	−200
1·81	1·76	+10,000	3,000
1·47	1·46	+20,000	6,000
1·19	1·22	+30,000	9,000

$T=11$°C, barometer=779 mm. Hg.

Tables I. and II. have been inserted to show the agreement between experiment and calculation. The calculated values have been obtained from the ground level readings of the meter by the method given in the appendix. The index n' in equation (6), on which the calculations are based, has been taken as 0·5. It will be seen that the discrepancy between experiment and calculation is never large. The greatest error occurs in the second row of Table II.—viz., 1·81—and is probably an experimental one.

4. *Experiments at Low Temperatures and Constant Pressure.*

The apparatus used for these tests is shown diagrammatically in Fig. 8. The gas from an oxygen cylinder passes through throttle valve (1) and thence to meter A. The water manometer is used to measure the fall in pressure between meters

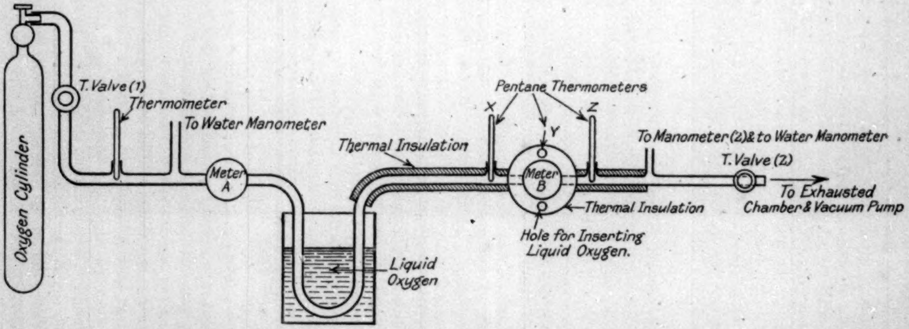
Oxygen Cylinder
T. Valve (1)
Thermometer
To Water Manometer
Meter A
Thermal Insulation
Liquid Oxygen
Pentane Thermometers
X
Y
Z
Meter B
Thermal Insulation
Hole for Inserting Liquid Oxygen.
To Manometer (2) & to Water Manometer
T. Valve (2)
To Exhausted Chamber & Vacuum Pump

FIG. 8.—DIAGRAM OF APPARATUS USED FOR LOW TEMPERATURE TESTS.

A and *B*. Manometer (2) is used to read the pressure of the gas issuing from *B*. During the experiments, which were conducted at ground level, the pressure drop *AB* was always less than 3·5 cm. of water (about 2·5 mm. Hg.), so that it is sufficiently accurate to assume that the pressure of the gas is the same at both meters. Care must be taken in connecting the manometers to the main flow pipe. When T pieces are used they must be free from internal projections. A projection or lip on the connection to the main pipe will cause either suction or pressure, according to its position with reference to the gas stream. This will affect the manometer accordingly.

After passing through *A*, the gas traverses a U tube, thermally insulated from *A*, immersed in liquid oxygen, which produces a reduction in temperature. Thence it passes through *B*, which is situated in an enclosure packed with cotton wool or other insulating material, to an exhausted chamber, the pressure in which is kept approximately constant. If the flow-meter is to be tested at atmospheric pressure, the chamber and throttle valve (2) can be disconnected. By means of the apparatus of Fig. 8 it is possible to carry out experiments in which the gas pressure can be fixed at 760 mm. Hg., or at values greater or less than this. This is effected by manipulation of the throttle valves. In working at pressures different from that of the external atmosphere great care must be taken to make the whole of the apparatus leak free. The temperature is controlled by altering the length of the U tube in the liquid oxygen. It depends largely on the mass of gas flowing through the tube per minute. The larger the mass of gas up to a certain limit the lower the temperature of the gas passing through meter *B*. When the system is thoroughly cooled, the influence of the mass of gas is considerably less. The temperature control may be facilitated by an additional U tube immersed in water, which can be heated if necessary, or in liquid oxygen, according to requirements. It is necessary, when readings are taken at constant temperature, to wait until conditions are steady. The temperature of the gas should be equal on both sides of flow-meter *B*. To secure this condition it is essential that the flow-meter and the connections thereto, should be well insulated thermally. It was found necessary to pour liquid oxygen into the insulation surrounding the meter in order to attain the desired temperature condition. Before the flow-meter was insulated, thermometer *X* registered 95°C. and thermometer Z 10°C. The temperature of the gas issuing from the nozzle, however, was almost 95°C., and the results were not so erroneous as might have been expected. In experiments of this nature it is inadvisable to use air, owing to the solidification of the moisture therein, which chokes the system and causes a large pressure drop between meters A and B, apart from other effects. Some experiments were conducted by replacing the oxygen cylinder and valve (1) by a calcium chloride drying tower open at one end to the atmosphere. These were not satisfactory for the above reason. In this case a correction must be made for the drop in pressure in the chloride tower.

When an experiment is being performed with the above apparatus the temperature of the gas passing through flow-meter B is adjusted to a value, T_2, and the flow regulated so that the flow-meter gives a definite reading. The pressure at meter B is adjusted to a certain value, say the atmospheric pressure at the time, by manipulating the throttle valves. The reading of manometer (1) should be less than 3·5 cm. of water. When conditions are steady, readings are taken of T_1, flow-meter A, T_2, flow-meter B and the barometric height. The calibration of flow-meter A under conditions almost identical with those during the test is found by disconnecting the whole of the apparatus to the right of A, and substituting a wet meter, the gas from which is discharged into the atmosphere. A correction can be made for any variation in conditions, as shown hereafter. The readings of the wet meter are reduced to N.T.P. These represent the N.T.P. volumes of gas passed through flow-meter B during the experiments. It was found more accurate to mark the positions of the pointer of flow-meter A on a paper scale pasted to the glass than to read A directly, since the position of the needle during calibration could be obtained more accurately. Meter B is then calibrated in a manner similar to A, when the pointer indicates the same readings as those obtained at low temperatures. It is really immaterial, for purposes of comparison, under what conditions A and B are calibrated at ground level, so long as they are identical and the temperature is known.

5. *Results of Experiments at Low Temperatures and Constant Pressure.*

Some of the results obtained are shown in Table III. It appears that the agreement between experiment and calculation is better for the two higher than for the two lower readings. Considering the liability to errors of observation at the lower part of the instrument scale, there is a fair agreement between theory and experiment.

TABLE III.

$T=13°C.$, barometer=761 mm. Hg.

Temperature, T_2(°C.).	Reading of meter B (virtual litres per minute).	N.T.P. litres per minute passing through meters A and B.	
		Experiment.	Calculation, assuming $n'=0{\cdot}5$
—100	5·13	6·55	6·56
—90	4·18	5·28	5·22
—80	3·14	3·90	3·83
—70	2·33	2·82	2·77

6. *Experiments at Increased Pressures and Constant Temperature.*

From equation (6) viz., $V=c_3(p/T)^{0\cdot5}$, it is evident that the effect of increasing the pressure of the gas passing through the flow-meter, is equivalent to lowering the temperature, since both cause an increase in density. Similarly an increase in temperature is equivalent to a decrease in pressure, the density in this case being lowered. In view of this, some experiments were carried out at increased pressures and constant temperature. The apparatus was arranged as shown in Fig. 9. The increase in pressure above that of the external atmosphere is measured by a mercury manometer, and the reduction in pressure before the gas passes through A is effected by a throttle valve. As in the experiments at low temperatures, the reading of B is observed and the corresponding position of the pointer of meter A is marked on paper pasted to the glass top. The calibrations of both meters are determined as before. In order to avoid a difference in temperature of the gas at the two meters, due to expansion at the throttle valve, it may be necessary to use a heating

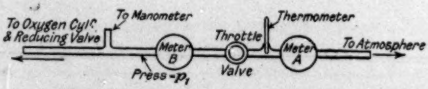

FIG. 9.—DIAGRAM OF APPARATUS USED FOR INCREASED PRESSURE TESTS.

tube between the throttle valve and meter *A*. The heating tube was not required in the present case, since the temperature was equal on the high and low pressure sides of the system, due to heat obtained from the connecting pipe, &c. In these experiments care must be taken to avoid leaks on the high pressure side of the system.

7. *Results of Experiments at Increased Pressures and Constant Temperature.*

TABLE IV.

$T=13^\circ$C., barometer=760 mm. Hg.

Pressure (p_1 mm. Hg.)	Temperature which produces same change in density as p_1 (°C.).	Reading of *B* (virtual litres per min.).	N.T.P. litres per minute passing through meters *A* and *B*.	
			Experiment.	Calculation.
1,255	−100	5·13	6·55	6·56
1,188	−90	4·18	5·28	5·22
1,125	−80	3·14	3·88	3·83
1,069	−70	2·33	2·72	2·77

The results are given in Table 4. The agreement between experiment and calculation is of a similar order to that obtained in the tests at low temperatures.

The meter readings for corresponding temperature and pressure, *i.e.*, equal gas densities, are tabulated in Table V. in order to show the agreement between the two sets of observations, both of which were made under practically the same atmospheric (external) conditions.

TABLE V.

$T=13^\circ$C., barometer=760 mm. Hg.

Virtual litres per minute on meter *B*.	N.T.P. litres per minute, the gas density being the same for both sets of figures (experiment).	
	Increased pressure.	Reduced temperature.
5·13	6·55	6·55
4·18	5·28	5·28
3·14	3·88	3·90
2·33	2·72	2·82

Experiments have not been carried out at low temperatures and low pressures (simultaneous conditions), nor under various other conditions possible. Unless extreme temperatures and pressures are used there is no good reason to doubt the validity of equation (6), under various conditions of temperature and pressure.

In conclusion it should be mentioned that the instrument described herein behaves quite well when fitted to an aeroplane. The accuracy required in practice is not greater than 5 per cent., and the readings are certainly more accurate than this. The vibration of an aeroplane when in flight has little influence on the accuracy. The reading is affected when the aeroplane has an angular acceleration about an axis parallel to that of the spindle of the instrument movement. It is not likely to be read, however, when the acceleration is large.

8. *Summary of Results.*

The chief results may be summarised as follows :—

1. The force normal to the flow-meter vane, for a limited angular movement, at various temperatures and pressures which prevail in the upper and lower strata of the atmosphere, is given by $F=k\rho Av^2$. The value of k varies with the position of the vane relative to the nozzle.

2. A flow-meter will only read correctly if the temperature pressure and density of the gas is the same as that used during calibration. A correction can be made, however, for variation in temperature and pressure by means of the formula, $V=V_0(T_0/p_0 \,.\, p/T)^{0.5}$, where V=N.T.P. volume of gas under new conditions, V_0=meter reading*; T_0 and p_0=calibration temperature and pressure; p and T=new temperature and pressure. When corrected for a gas of different density the formula becomes $V=V_0(\rho/\rho_0 \,.\, T_0/p_0 \,.\, p/T)^{0.5}$. (*See* Appendix for significance of symbols.)

APPENDIX.

Relation between Altitude and N.T.P. Volume of Gas per minute for a Definite Meter Reading.

The relation between the height of the isothermal atmosphere and the pressure is given by

$$h=a \log_e b/p,$$

where h=height, p=pressure, a and b=constants.

$$\therefore \qquad p=be^{-h/a}. \qquad \qquad (8)$$

---

* This is N.T.P. litres when the temperature of the gas is T_0 and its pressure p_0.

From (6) $V=c_3(p/T)^{n'}$, where V=N.T.P. volume.

Equating the values of p from (6) and (8) we get

$$V=\frac{d}{T}e^{-fh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $d=a$ constant,

and $f=n'/a$.*

If n' and T are constant for a given reading of the meter, the relation between V and h is an exponential one [from (9)]. When, for all values of h, the product fh is small compared with unity, the relation between V and h is approximately linear. This can be shown by expanding e^{-fh} thus :—

$$V=\frac{d}{T}e^{-fh}=\frac{d}{T}\{1-fh+f^2h^2/2\,!-f^3h'^3/3\,!+f^4h^4/4\,!-, \&c.\}.$$

If $fh=0{\cdot}1$, say, all the terms beyond the second are negligible, and we obtain

$$V=\frac{1}{T}(d-dfh), \text{ approximately,}$$

or

$$V+mh=\frac{d}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $m=\dfrac{df}{T}$. (*See* (7).)

Taking the average value of n' as 0·5, and the greatest value of h as 40,000 feet, $fh=\dfrac{0{\cdot}5\times 40{,}000}{27{,}200}=0{\cdot}74$. In this case the first five terms of the above series are important, and the relation between V and h, over the range $h=0$ to $h=40{,}000$ feet=12,200 metres, cannot be taken as linear. The range from ground level, over which (10) is applicable is 7,000 feet=2,130 metres.

If, for a definite meter reading, the value of n' varies with the altitude, it is evidently some function of h. Thus we may write $V=\dfrac{d}{T}e^{\phi h}$, where $\phi(h)$ is a function of h. It is probably sufficiently accurate for practical purposes to assume that n' is constant.

---

* $f=1{\cdot}84\times 10^{-5}$ when h is in feet, and $5{\cdot}56\times 10^{-5}$ when h is in metres.

Correction of Meter Reading at Various Altitudes.

In (9), when $h=0$, $V=V_0=d/T$=meter reading at the ground. Since $T=283°$abs., $d=283\ V_0$, and hence

$$V=283V_0\,.\,e^{\gamma h}/T. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If there was no variation in V with altitude and temperature, equation (11) would read $V=V_0$. Therefore, the quantity $283e^{\gamma h}/T$, may be regarded as the correction factor of the meter corresponding to altitude h and absolute temperature T (it is assumed that V_0 corresponds to the N.T.P. volume when calibration is effected at 10°C. and 760 mm. Hg.). The absolute temperature can be written in terms of h (approximately), thus : $T=283-1{\cdot}75\times10^{-3}h$, where $1{\cdot}75\times10^{-3}$ is the temperature gradient per foot of ascent. Equation (11) now becomes

$$V=\{283V_0/(283-1{\cdot}75\times10^{-3}h)\}\,.\,e^{\gamma h}=V_0y, \quad . \quad (12)$$

where $\quad y=\{283/(283-1{\cdot}75\times10^{-3}h)\}\,.\,e^{\gamma h},$

$\quad =$correction factor.

Curves are given in Fig. 10, showing the correction factor at various altitudes (*a*) when the temperature is constant, *i.e.*, in the isothermal atmosphere ; (*b*) when the variation of temperature with altitude is taken into account. The difference between curves (*a*) and (*b*) is due to the compensation caused by reduction in temperature. In curve (*b*) an average value of 1·75°C. has been taken as the fall in temperature per 1,000 ft.=300 metres of ascent. Fig. 11 shows the curves of Fig. 10 on a pressure base.

Correction of Meter Reading Due to Variation in Temperature and Pressure at Ground Level or at any Altitude.

From (6), $V=c_3(p/T)^{0{\cdot}5}$; where V=N.T,P. volume at pressure p and absolute temperature T. When $p=p_0$, $T=T_0$ and $V=V_0$=meter reading, these being conditions during calibration.

Thus, substituting these values in (6), we find

$$c_3=V_0(T_0/p_0)^{0{\cdot}5}\,;$$

$$\therefore \qquad V=V_0(T_0/p_0\,.\,p/T)^{0{\cdot}5}, \quad . \quad . \quad . \quad . \quad (13)$$

where V=N.T.P. volume.

By means of (13) it is possible to correct the meter reading to any absolute temperature T and pressure p.

Example of the Use of Correction Factor.

In the curves of Figs. 10 and 11 it has been assumed that the temperature of the gas used for calibration purposes is 10°C.

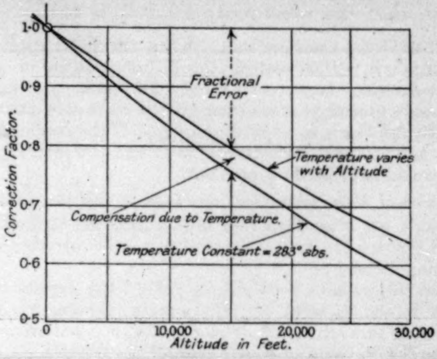

Fig. 10.—Curves showing Correction Factor of Meter at any Altitude.

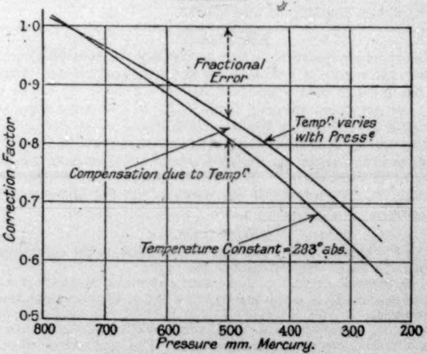

Fig. 11.—Curves of Fig. 10 on a Pressure Base.

and its pressure 760 mm. of mercury. These conditions will not be attained in general. Suppose the meter reads 4 litres per minute when calibration conditions are such that $T_0=15$°C.

and $p_0=737$ mm. Hg., *i.e.*, the N.T.P. volume of gas passing through the meter per minute under these conditions is 4 litres. At 10° C. and 760 mm. Hg. the N.T.P. volume of gas when the meter reads 4 L.P.M. is $4\left(\frac{288}{737}\times\frac{760}{283}\right)^{0\cdot5}$ $=4\times1\cdot025\risingdotseq4\cdot1$ litres per min. When the meter reads 4 at, say, 30,000 ft.=9,000 metres, the N.T.P. volume of gas is $(4\cdot1\times\text{correction factor})=4\cdot1\times0\cdot64\risingdotseq2\cdot6$ litres per minute. The above procedure is necessary if the correction factors of Figs. 10 and 11 are to be used properly. If the reading of the meter had not been corrected to 10°C. and 760 mm. Hg., the error would have been 2·5 per cent.

Correction of Meter when Used for Gases of Different Densities.

When a gas other than that used during the calibration is passed through the meter, the reading can be corrected in the following manner :—

From (13) we have $V=V_0(T_0/p_0\ p/T)^{0\cdot5}$ the density of the gas at T_0 and p_0 being ρ_0. If, however, the density of the gas flowing through the meter at T_0 and p_0 is ρ, the true value, *i.e.*, the N.T.P. value, of V_0 is $V_0\,(\rho/\rho_0)^{0\cdot5}$. Whence

$$V=V_0(\rho/\rho_0\,.\,T_0/p_0\,.\,p/T)^{0\cdot5},$$

where V_0 is the reading of the meter.

ABSTRACT.

The theory of an instrument for measuring the rate of flow of a gas is outlined, the effects of variation in the temperature and pressure of the gas being taken into consideration. This theory is tested experimentally for pressures varying from 1,250 to 250 mm. Hg., and for temperatures from 10°C. to 100°C. It is found to be fairly accurate. The results are applied to the measurement of the rate of flow of gas on an aeroplane in the upper atmosphere, where a reduction in temperature and pressure is encountered. It is shown that the instrument reading for a certain N.T.P. volume of gas depends on the altitude, but that this volume can be obtained by using a correction factor.

DISCUSSION.

Mr. C. C. **Paterson** asked if the results depended on the structure of the meter, since the Paper only referred to one type.

Dr. A. **Griffiths** suggested that some automatic method of compensating the error would be advantageous as a direct reading instrument was always desirable.

The **Author**, in reply, said that compensating devices were under investigation when the war came to an end. The pressure effect could be compensated by using an aneroid control to vary the area of the nozzle, but he did not see how the temperature effect could be easily corrected. As regards the structure of the meter, the diameter of the nozzle was about 3/32nds of an inch, and the rectangular vane was mounted very similarly to the moving parts of a moving coil ammeter. Either electromagnetic damping or, for laboratory purposes, ordinary fluid damping could be used.

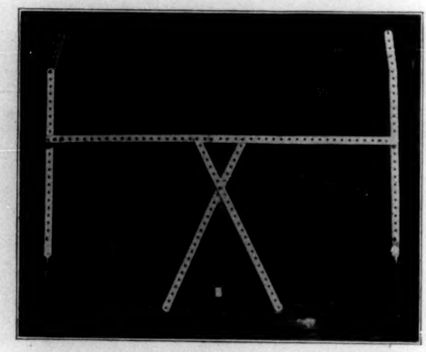

FIG. 1.—COMPLETE BALANCE, FILTER PAPER IN POSITION.

To face p. 21.]

II. *A Cheap and Simple Micro-Balance.* By Capt. J. H. Shaxby, *B.Sc., University College, Cardiff.*

Received, July 11, 1919.

The name micro-balance should, perhaps, hardly be applied to the instrument described below, as it is designed for maximum loads varying between 5 milligrams and 5 decigrams. It is a simplified and rough form of a piece of apparatus designed for smaller loads, to be used in a research which the war has postponed.

The present type was put together for bacteriological use, to weigh small drops of fluid absorbed in bits of filter paper, rapidly and with an accuracy of one or two parts in 1,000. The balance was required to be reasonably robust and portable, and above all cheap. It has proved suitable for its purpose, and also in weighing small samples of mixed powders.

The principle is very simple: A light thread has its extremities attached to the lower ends of a pair of nearly vertical rods 2 ft. or so apart, each of which is pivoted at a point very little above its centre of mass. Thus, a very light load suspended at the middle of the thread causes a considerable depression of that point. The depression is not quite proportional to the load; but a series of five weights, one-fifth, two-fifths, &c., of the maximum load, gives means of calibration.

Detail of construction is illustrated in the diagrams, and is as follows: The metal parts are built up of "Meccano" strips of various lengths, fixed together where necessary by the small nuts and bolts supplied with the sets. The fixed framework consists of a long horizontal arm, 2 ft. in length, built up of two 25-hole strips, with three holes overlapping, supported at a height of about 1 ft. by a sort of tripod. This is built up of three 25-hole strips (as shown in the photograph, Fig. 1), two of which are in a vertical plane, while the third, bent slightly about the third to fifth holes from the top, slopes away like the back leg of some easels. The lower end of this leg may be bent so as to lie flat on the table, to which a drawing-pin through the last hole clamps it rigidly and holds the whole frame steady.

To the ends of the horizontal beam are fixed short pieces bent twice at right angles, so that they lie as shown in the plan of the end of the beam (Fig. 2).

Each of the two vertical suspended arms consists of two strips, the lower a 25-hole one and the upper a 6-hole one, bolted together through their end holes. The purpose of this is to allow of ready variation of sensitiveness, as explained later. A short piece ($\frac{1}{4}$ in. long) of glass tubing with rounded-off ends is cemented into the 15th hole from the free (lower) end of the longer strip; this hole is slightly below the centre of mass of the compound strip. A small nut and bolt a few holes lower brings the centre of mass down sufficiently and permits of adjustment. The middle points of two short bits of copper wire are joined by a thread of ordinary sewing silk, so that the free length of thread is about equal to the distance *AB* (Fig. 2). One bit of wire is then passed through the end hole at *A*, through one of the glass tubes, *T*, and through the end hole at *B* (bent temporarily toward *A*), so that the wires now lie flat on the outer sides of *A* and *B*, and keep the thread taut.

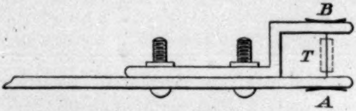

FIG. 2.—PLAN OF END OF FIXED BEAM.

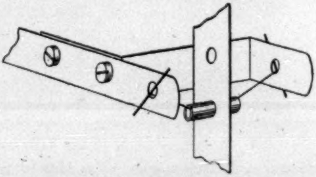

FIG. 3.—DIAGRAM OF SUSPENSION.

The movable arm is thus suspended (Fig. 3) with very little friction. The other arm is supported similarly at the other end of the fixed horizontal beam. Light aluminium vanes attached to the lower ends of the vertical arms play in water dashpots and keep the whole dead beat.

A thin wire hook is also soldered on near the bottom of each arm (Fig. 4). Thin wire loops are attached to the ends of an unspun silk or other fine fibre of length equal to the distance between the pivots of the movable arms, the loops are slipped over the hooks and the lower ends of the movable arms thus linked together. The tension in the thread, and thus the sensitiveness of the balance, can be largely and immediately varied by changing the angle between the two strips which make up each compound movable arm. The parts of the balance will not in general be quite symmetrical, so that if one arm has its two parts in a straight line the other will have to be adjusted slightly, in order that the lower portions may rest at the same small angle with the vertical, *i.e.*, so that the middle point of the fibre shall lie directly under the middle point of the fixed horizontal beam. To decrease the sensitiveness, both the upper short strips are to be turned inward. If they are turned outward the fibre is usually no longer kept taut, so greater sensitiveness cannot be obtained thus ; if it is necessary, the small adjusting nuts are raised nearer to the pivots.

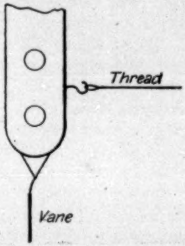

FIG. 4.—LOWER END OF MOVABLE ARM.

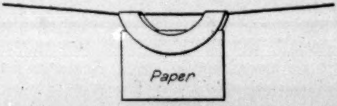

FIG. 5.—CLIP AND FILTER PAPER.

The loads are supported by a very light aluminium-foil clip loosely looped over the middle of the fibre ; the scrap of filter paper is inserted in this (Fig. 5). For powders a small thin paper scalepan replaces the clip.

Weighings are made thus. The calibrating weights are inserted in turn, and the resulting depressions noted. In the case of weighing a drop the filter paper is included in all the observations. The depression due to the object to be weighed is finally observed, and its weight readily found from a calibration curve.

Of course the deflections could easily be measured by a cathetometer microscope, but as has been said above, cheapness was a *sine qua non*. The balance itself costs about eighteenpence, and an auxiliary microscope costing several pounds was clearly ruled out. The deflections were measured as follows: A cardboard slider carrying a zero mark, and pierced by a circular "peephole" about the size of the pupil of the eye is attached to a millimetre scale, which is set up vertical 2 ft. or 3 ft. in front of the balance. A small cardboard tube closed at one end (the cap of an old rather wide thermometer case) was blackened inside, and across its mouth was stretched a fine fibre, which thus showed up sharp and white against the dark background of the interior of the tube. This was put an inch or two behind and slightly lower than the middle of the long balance fibre. If both fibres are horizontal a position can be found for the slider on the scale, so that one fibre hides the other from an eye looking through the peephole, when the clip is unloaded, and for each successive load. The deflections thus measured on the vertical scale are clearly equal large multiples of the actual deflections of the middle of the balance fibre. In practice it is found that the setting of the slider can be made with the greatest certainty, not with a horizontal fibre or a pair of cross-hairs on the tube, but with a single fibre set at a slight angle with the horizontal. The slider is adjusted so that the horizontal part of the balance fibre appears to bisect the short fibre in the tube. The angle between the two fibres should be small, and the slider can then be set with great precision; an astonishingly small displacement noticeably upsets the bisection.

ABSTRACT.

The instrument, devised for bacteriological use, had to be cheap and moderately robust. It consists of a long horizontal fibre joining the lower ends of two vertical beams, each pivoted very little above its centre of mass. A small weight acting at the middle of the fibre thus causes a considerable depression. This is read off by arranging a slider on a vertical millimetre scale about 2 ft. in front, so that the middle of the fibre and a second short fibre placed just behind it are in line with a "peephole" on the slider. Adjustment is provided for quickly and largely altering the sensitiveness. The deflections are converted to masses by the use of calibrating weights. The apparatus is built up from a "Meccano" set.

DISCUSSION

Mr. C. C. **Paterson** thought the instrument would have many useful applications. Had the author considered the possibility of making it a null method ? A simple electromagnetic arrangement could be employed for the purpose.

Prof. **Bragg** suggested a simpler null method, which consisted in the addition of weights to attachments to the side pieces. The arrangement could be such that quite considerable weights had to be added to produce a slight disturbance of the balance.

Mr. T. **Smith** asked if the author had tried reading the position of the fibre against a thick thread mounted parallel to it, instead of the inclined thread he had described. If the thick thread were brightly illuminated, and the fibre adjusted so that it appeared to bisect the width of the thread a very accurate setting would result.

Dr. D. **Owen** asked for the sensibility of the instrument shown at the meeting.

Mr. F. J. **Harlow** asked the form of the calibration curve.

Prof. **Lees** said the instrument reminded him of a well-known text-book method for measuring small horizontal forces, in which the force produced a small horizontal displacement of a weight suspended by a long thread. If two such suspended weights were connected by a horizontal fibre to which small loads could be attached, a micro-balance somewhat similar to Mr. Shaxby's would be obtained.

The **Author**, in reply, said Mr. Paterson's suggestion was interesting. He had made some attempts to convert the instrument into a null one with a magnetic control below the load. He had also tried Prof. Bragg's suggestion, by mounting weights on short horizontal arms near the fulcrums of the vertical rods. As regards sensitivity, readings could be repeated to about 1/500th of the maximum load for which the balance was adjusted. The calibration curve was nearly a straight line.

III. *The Resolution of a Curve into a Number of Exponential Components.* By JOHN W. T. WALSH, *M.A., M.Sc.* (*From the National Physical Laboratory.*)

RECEIVED SEPTEMBER 9, 1919.

Synopsis.—In many problems connected with the growth and decay of radio-active substances it is necessary to resolve an observed curve into two or more component exponentials of different half-periods.

Thus, for example, in a series of n successive transformations the amount of the nth product $N(t)$ at any time t is given by

$$N(t)=c_1e^{-\lambda_1 t}+c_2e^{-\lambda_2 t}+\ldots\ldots+c_ne^{-\lambda_n t}(*).$$

The problem dealt with in this Paper is the determination of the $2n$ constants $c_1\,c_2\,\ldots\,c_n$ and $\lambda_1\,\lambda_2\,\ldots\,\lambda_n$ from the form of the observed curve for $N(t)$.

Let it be assumed that the expression for the curve to be resolved has the form

$$B=a_1e^{-\lambda_1 t}+a_2e^{-\lambda_2 t}+\ \ldots\ +a_ne^{-\lambda_n t},$$

and let $B_0\,B_1\,B_2\,\ldots\,B_{2n-1}$ be $2n$ values of B at equidistant intervals of t. Then, assuming $a_1\,a_2\,\ldots$ to contain the constants $e^{-\lambda_1\tau}\,e^{-\lambda_2\tau}\,\ldots$ where τ is the starting point of the time series, and writing $a_1a_2\,\ldots$ for $e^{-\lambda_1 t}$, $e^{-\lambda_2 t}\,\ldots$

$$B_0=\Sigma a_1,$$
$$B_1=\Sigma a_1a_1,$$
$$B_2=\Sigma a_1a_1^2,$$
$$\ldots\ldots$$
$$B_{2n-1}=\Sigma a_1a_1^{2n-1}.$$

If now $a_1a_2\,\ldots\,a_n$ be the n roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ \ldots\ +p_{n-1}x+p_n=0,$$

then

$$B_{(n+m)}+p_1B_{(n+m-1)}+\ \ldots\ +p_nB_m=0,$$

where $m<n$, for the left-hand member of this equation

$$=\Sigma a_1a_1^{n+m}+p_1\Sigma a_1a_1^{n+m-1}+\ \ldots\ +p_n\Sigma a_1a_1^{m},$$
$$=\Sigma a_1[a_1^{n+m}+p_1a_1^{n+m-1}\quad+\ \ldots\ +p_na_1^{m}],$$

which is zero, for since $a_1a_2\,\ldots\,a_n$ are the roots of the equation in x written above, each term written in brackets vanishes separately. Hence we have n equations of the type

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* Rutherford, "Radioactive Substances and Their Radiations." 1913, p. 422.

$$B_{n+m}+p_1B_{n+m-1}+\ .\ .\ .\ +p_nB_m=0,$$

where m has the n integral values 0 to $(n-1)$.

Hence, eliminating $p_1p_2\ldots p_n$ between these equations and the equation in x written above, we have

$$\begin{vmatrix} x^n & x^{n-1} & x^{n-2}\ldots\ldots & x & 1 \\ B_n & B_{n-1} & B_{n-2}\ldots\ldots & B_1 & B_0 \\ B_{n+1} & B_n & B_{n-1}\ldots\ldots & B_2 & B_1 \\ \cdot & \cdot & \cdot\quad\cdot\quad\cdot & \cdot & \\ B_{2n-1} & B_{2n-2} & B_{2n-3}\ldots\ldots & B_n & B_{n-1} \end{vmatrix}=0.$$

From this equation the values of $\alpha_1\alpha_2\ldots\alpha_n$ are found (graphically if necessary).

Substituting in the original equations the values of $\alpha_1\alpha_2\ldots$ thus obtained, the values of $a_1a_2\ldots$ are determined by means of any convenient set of n of the original equations for $B_0B_1\ldots.$

As an example of analysis into two exponentials let us take a case in which the observed values at 100, 400, 700 and 1,000 days were 278, 107, 70 and 50 units respectively.

Then

$$B_0=278$$
$$B_1=107$$
$$B_2=\ 70$$
$$B_3=\ 50$$

and

$$\begin{vmatrix} x^2 & x & 1 \\ 70 & 107 & 278 \\ 50 & 70 & 107 \end{vmatrix}=0$$

i. e.,

$$8{,}011x^2-6{,}410x+450=0,$$

giving

$$x=0{\cdot}7014 \quad\text{and}\quad 0{\cdot}0769.$$

Hence

$$a+b=278$$

and

$$0{\cdot}7014a+0{\cdot}0769b=107$$
$$0{\cdot}6245a=107-21{\cdot}4=85{\cdot}6,$$

$\therefore$

$$a=137{\cdot}1,$$

and

$$b=140{\cdot}9.$$

Hence

$$B=137{\cdot}1\times(0{\cdot}7014)^{\frac{t-100}{300}}+140{\cdot}9\times(0{\cdot}0769)^{\frac{t-100}{300}}$$
$$=154{\cdot}3\,e^{-0{\cdot}00118t}+331{\cdot}4\,e^{-0{\cdot}00855t},$$

where t is expressed in days, from the point $t=0$.

If, as is usual, the observations are taken over an extended period, possibly at irregular intervals of time, the values of $B_0 B_1 \ldots$ required by the above method are obtained by plotting the observed values, drawing a curve through them by eye, and then reading from the curve the values at the time interval selected.

It is, however, desirable to ensure that the values of the constants found represent the *whole* of the observed values so as to make the probable error a minimum. This may be done by finding, from the calculated expression, the value of the function at each observed point of the curve. If ∂_r be the difference between the calculated and the observed values at any point, and if $a_1 \ldots a_n$ and $\lambda_1 \ldots \lambda_n$ be the values of a and λ found as above, then if the most probable values of these quantities be $a_1(1+\theta_1) \ldots$ and $\lambda_1(1+\varphi_1) \ldots$, θ and φ can be found from all the available equations of the type

$$\partial_r = \Sigma a_1(1+\theta_1)e^{-\lambda_1(1+\phi_1)t} - \Sigma a_1 e^{-\lambda_1 t}.$$

Now if θ and φ are small, so that squares and products may be neglected, this becomes

$$\partial_r = \Sigma \{a_1 e^{-\lambda_1 t}\theta_1 - a_1 \lambda_1 t \varphi_1\},$$

and from all the available observations, the most probable values of $\theta_1 \ldots$ and $\varphi_1 \ldots$ may be found by forming the normal equations in these quantities as usual.

As an example, seven of the observations from which the curve was drawn to give the values used in the example on p. 27 were as follows :—

TABLE I.

t.	B_t.	∂r.	B_t (calc.).	∂_r'.
41	402	+21·3	398·7	+3·3
74	325	+ 7·5	323·6	+1·4
139	224	− 7·8	227·3	−4·9
295	126	− 9·5	129·8	−3·8
443	99	+ 0·1	96·3	+2·7
669	71	− 0·3	70·3	+0·7
1,205	42	+ 4·6	36·9	+5·1

Using the expression found for B_t on p. 27, we can obtain the values of ∂_r given in the third column above. From these values the following normal equations result :—

$$85266\ \theta_1 + 76129\ \theta_2 + 19960\ \varphi_1 + 53905\ \varphi_2 + 2295 = 0$$
$$76129\ \theta_1 + 96588\ \theta_2 + 7444\ \varphi_1 + 52912\ \varphi_2 + 5260 = 0$$
$$19960\ \theta_1 + 7444\ \theta_2 + 10253\ \varphi_1 + 8871\ \varphi_2 - 51{\cdot}9 = 0$$
$$53905\ \theta_1 + 52912\ \theta_2 + 8871\ \varphi_1 + 38912\ \varphi_2 + 1015 = 0$$

These give

$$\begin{aligned}\theta_1&=+0{\cdot}0039\\ \theta_2&=+0{\cdot}134\\ \varphi_1&=+0{\cdot}0063\\ \varphi_2&=+0{\cdot}148_5\end{aligned}$$

Hence $$B=154{\cdot}9\,e^{-0{\cdot}00119t}+375{\cdot}8\,e^{-0{\cdot}00982t}.$$

The values of B_t found from this expression, and the difference from the observed values ∂_r', are shown in columns 4 and 5 respectively of Table I.

The method is, of course, analogous to the Lagrange-Dale method of analysis of a compound periodic function,* simplified for the less complicated case of pure exponentials.

It is clear that a similar method may be employed to analyse a curve of the form $Q=\Sigma a_1 e^{-\lambda_1 t}\cos(p_1 t+\alpha_1)$, for this is identical with the form

$$Q=\Sigma\{A_1 e^{-l_1 t}+B_1 e^{-m_1 t}\}.$$

APPENDIX.

The following method of computing the determinants of the fourth order required for the treatment of a compound exponential of two elements may be of interest. If a determinant of the fourth order be

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\\ c_1 & c_2 & c_3 & c_4\\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

it may be written as

$$\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}\times\begin{vmatrix} c_3 & c_4\\ d_3 & d_4\end{vmatrix}+\begin{vmatrix} a_3 & a_4\\ b_3 & b_4\end{vmatrix}\times\begin{vmatrix} c_1 & c_2\\ d_1 & d_2\end{vmatrix}-\begin{vmatrix} a_1 & a_3\\ b_1 & b_3\end{vmatrix}\times\begin{vmatrix} c_2 & c_4\\ d_2 & d_4\end{vmatrix}$$

$$-\begin{vmatrix} a_2 & a_4\\ b_2 & b_4\end{vmatrix}\times\begin{vmatrix} c_1 & c_3\\ d_1 & d_3\end{vmatrix}+\begin{vmatrix} a_1 & a_4\\ b_1 & b_4\end{vmatrix}\times\begin{vmatrix} c_2 & c_3\\ d_2 & d_3\end{vmatrix}+\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}\times\begin{vmatrix} c_1 & c_4\\ d_1 & d_4\end{vmatrix}$$

Thus the 5 fourth order determinants in the solution of the four normal equations may be expressed as sums of products of 20 minors of the second order.

The sixth order determinant may be similarly resolved either into second or third order minors. If third order minors are used, work with logarithms may be facilitated by the following scheme of writing :—

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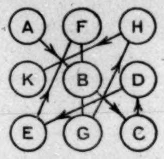
* J. B. Dale, Roy. Astron. Soc., M.N., 74, p. 628, May, 1914.

Write $\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix}$ as



where Ⓐ stands for log A, etc.

Adding each row and column, and writing down the anti-logarithms, the difference between the sum of the anti-logarithms of the rows and the sum of the anti-logarithms of the columns is the value of the determinant, for this

$$=AEK+BFG+CDH$$
$$-GEC-HFA-KDB.$$

ABSTRACT.

The Paper gives a method for the resolution of a curve of the compound exponential form $B=\sum_1^n a_1e^{-\lambda_1 t}$ into its components, the values of a and λ for the n different exponential terms being found from $2n$ values of B equidistant along the axis of t. A method is also given for finding the most probable values of these constants from any number ($>2n$) of observed values of B taken at irregular intervals of t.

DISCUSSION.

Mr. T. **Smith** said the crux of the matter appeared to lie in the proper choice of the number of components, and this was not touched on in the Paper. It was possible to come to a decision provided the observations covered a sufficiently long period. Suppose one felt justified in taking n terms, but was uncertain whether or not to go to $n+1$, a comparison of the outstanding errors when first n and then $n+1$ terms were taken would reveal whether there was any advantage in introducing the extra one.

Prof. **Bragg** asked whether in the event of uncertainty as to taking n or $n+1$ terms it made much difference which was adopted. Did all the old λ's change and another one of comparable magnitude come on ?

Mr. T. **Smith** said the new λ might well be quite large.

Mr. E. H. **Rayner** pointed out that it depended on the relative values of a, and λ whether there would be much difference in taking n or $n+1$.

Prof. **Lees** asked if the method worked when the a's were negative ?

The **Author**, in reply, said that in some curves the value of n was known from a priori considerations ; and in some other cases it was sufficient to take enough terms to fit the experimental curve well. The modifications of the λ's in taking $n+1$ instead of n terms was not simple. In ordinary cases all the λ's would be varied, and a new one comparable with the others added. The method applied equally to negative a's.

IV. *On the Self-inductance of Single-layer Flat Coils.* By S. BUTTERWORTH, *M.Sc., The National Physical Laboratory.*

RECEIVED OCTOBER 1, 1919.

1. The extended use of flat inductance coils in high-frequency electrical measurements renders desirable the development of simple formulæ for the predetermination of the inductance of such coils. A modification of Weinstein's formula* for coils of rectangular channel section has usually been employed for this purpose. This formula fails, however, when the inner radius is small.

It is the purpose of the present Paper to extend the modified Weinstein formula, to supply a formula suitable for coils of small inner radius, and to use the two formulæ to compute a table (Table I.) for the use of designers of flat inductance coils. Some applications of this table are also given.

2. The method of obtaining the formulæ is one of integration, starting with a suitable formula for the mutual inductance between two coaxial co-planar circles.

Maxwell† has given two elliptic integral formulæ for the mutual inductance between two coaxial circles. When the circles are also co-planar these reduce to

$$M=4\pi\sqrt{Aa}\,\{(2/k-k)K(k)-2/kE(k)\}, \quad . \quad . \quad . \quad . \quad (1)$$

or,

$$M=8\pi A\,\{K(a/A)-E(a/A)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which K and E are complete elliptic integrals of the first and second kind respectively ; a, A are the radii of the circles, and

$$k^2=4Aa/(A+a)^2.$$

In (2) $A>a$ always.

3. *Derivation of Extended Weinstein Formula.*

Let R be the mean radius of the flat coil, $2X$ be the coil depth, and n be the number of turns per unit of length. Also let $m(x, r)$ be the mutual inductance between two co-planar coaxial circles, whose mean radius is r and radius difference is $2x$. Then by summation of the mutual inductance between the elementary filaments of the flat coil the self-inductance of the latter is

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* Weinstein, "Wied. Ann.," XXI., p. 329, 1884 ; Bull. Bureau of Standards, VIII., p. 137, 1912.

† "Electricity and Magnetism," Vol. II., 701.

$$L=4n^2\int_0^X dx\int_{R-X+x}^{R+X-x} m(x,r)dr. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)^*$$

The integral (3) may be evaluated by expressing m in terms of x and r. The required expression may be derived from (1) by developing into a series involving only k' the complementary modulus of k, for

$$k'=(1-k^2)^{\frac{1}{2}}=(A-a)/(A+a)=x/r.$$

The suitable series† is

$$M=4\pi r\left(\Phi_0+\frac{1^2}{2^2}k'^2\Phi_1+\frac{1^2.1^2}{2^2.4^2}k'^4\Phi_2+\frac{1^2.1^2.3^2}{2^2.4^2.6^2}k'^6\Phi_3+\ldots\right) \qquad (4)$$

in which

$$\left.\begin{aligned}\Phi_0&=\log_e 4/k'-2\\ \Phi_1-\Phi_0&=\frac{1}{1}+\frac{2}{1}\\ \Phi_2-\Phi_1&=\frac{1}{2}-\frac{2}{1}\\ \Phi_3-\Phi_2&=\frac{1}{3}-\frac{2}{3}\\ .\quad .\quad &.\quad .\quad .\\ \Phi_n-\Phi_{n-1}&=\frac{1}{n}-\frac{2}{2n-3}\end{aligned}\right\}.$$

Applying (3) to the first four terms of (4)

$$L=16\pi n^2RX\left\{(\lambda-\tfrac{1}{2})+\tfrac{2}{3}z^2\left(\lambda+\frac{43}{12}\right)+\frac{44}{45}z^4\left(\lambda+\frac{96}{55}\right)\right.$$
$$\left.+\frac{412}{105}z^6\left(\lambda+\frac{98579}{86520}\right)+\ .\ .\ .\right\} \qquad (A)$$

in which

$$\lambda=\log_e 4R/X,\ z=X/4R.$$

The first two terms of this series constitute the modified Weinstein formula.

The series converges for all possible values of X and R.

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* For the method of building up this integral, *see* Butterworth, "Proc." Phys. Soc., XXXVII., p. 376, 1915.

† Bromwich, "Quarterly Journal of Pure and Applied Mathematics," No. 176, p. 363, 1913; Butterworth, "Phil. Mag.," XXXI., p. 276, 1916.

In the worst case when the inner radius is zero $X/R=1$, and the value of the terms in { } of (A) are then

$$0{\cdot}886 \qquad 0{\cdot}207 \qquad 0{\cdot}012 \qquad 0{\cdot}002.$$

Thus, assuming the four terms evaluated approximate sufficiently closely to the true sum, the Weinstein formula gives a result about 1 per cent. low in the extreme case.

To verify this result a formula is now developed which is suitable for large values of X, that is, for a small inner radius of the flat coil.

4. Using Maxwell's second formula {(2) above} and putting G for $K-E$, the flux Φ through a circle of radius x in the plane of the coil is

$$8\pi n\left\{x\int_{r_1}^{x} G(a/x)da+\int_{x}^{r_2} aG(x/a)da\right\},$$

in which r_1, r_2 are the inner and outer radii, or by a change of variables

$$\Phi=8\pi nx^2\left(\int_{r_1/x}^{1} G(\mu)d\mu+\int_{x/r_2}^{1} G(\mu)d\mu/\mu^3\right). \quad . \quad . \quad . \qquad (5)$$

The self-inductance of the flat coil is

$$L=n\int_{r_1}^{r_2} \Phi dx,$$

so that, integrating (5) by parts

$$L=\frac{16}{3}\pi n^2 r_2{}^3\int_{a}^{1}(1-a^3/\mu^3)G(\mu)d\mu, \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

in which $\qquad a=r_1/r_2.$

The evaluation of (6) involves integration of K and E with respect to the modulus. The required integrations have been discussed by the author in an earlier Paper.*

Using the notation and results obtained therein, (6) reduces to

$$L=\frac{8}{3}\pi n^2 r_2{}^3\{u_1-1-\alpha^3(1+v)-u+2\alpha E-\alpha(1-\alpha^2)K\}, \quad . \quad . \qquad (7)$$

in which

$$u=\int_{0}^{a} Kd\mu,\ v=\int_{a}^{1} Kd\mu/\mu,$$

$$u_1=\int_{0}^{1} Kd\mu=1{\cdot}831931248 \quad . \quad . \quad .$$

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* "Phil. Mag.," XXIX., p. 584, 1915.

(7) may be expressed in series by making use of the series for K, E, u, v, giving

$$L=\frac{4}{3}\pi^2n^2r_2{}^3[2(u_1-1)/\pi+\alpha^3\{2(u_1-1)/\pi+\tfrac{1}{6}-\log 4/\alpha+\sigma\}] \quad (B)$$

in which

$$\sigma=6\sum_1^\infty\left(\frac{1\,.\,3\,.\,5\,\ldots\,2n-1}{2\,.\,4\,.\,6\,\ldots\ldots\,2n}\right)^2\frac{2n+1}{2n(2n+1)(2n+3)}\alpha 2n.$$

This formula is most convergent when α is small and is thus suitable for coils having a small inner radius.

5. For the case of zero inner radius formula (B) gives

$$L=6{\cdot}96957n^2r_2{}^3,$$

formula (A) gives

$$L=6{\cdot}956n^2r_2{}^3,$$

while the first two terms only give

$$L=6{\cdot}87n^2r_2{}^3.$$

Thus, in the most unfavourable case for formula (A) the error is of the order 0·2 per cent., while the Weinstein formula gives a result 1·5 per cent. below the true value. For the case $\alpha=0{\cdot}5$

$$L=4{\cdot}743502n^2r_2{}^3 \text{ by } (A),$$

$$L=4{\cdot}743500n^2r_2{}^3 \text{ by } (B).$$

The two formula thus check each other.

For general computation formula (A) usually the most rapid to work with, while formula (B) may be used as a check formula.

6. For practical calculations we may write

$$L=Qn^2r_2{}^3, \quad \ldots\ldots\ldots\ldots \quad (C)$$

and tabulate Q as a function of r_1/r_2. This is done in Table I.

TABLE I.

r_1/r_2.	Q.	r_1/r_2.	Q.	r_1/r_2.	Q.
0·00	6·970	0·35	5·996	0·70	2·528
0·05	6·964	0·40	5·632	0·75	1·946
0·10	6·930	0·45	5·213	0·80	1·397
0·15	6·845	0·50	4·743	0·85	0·8892
0·20	6·728	0·55	4·231	0·90	0·4574
0·25	6·544	0·60	3·682	0·95	0·1394
0·30	6·300	0·65	3·105	1·00	0·0000

Self-inductance in centimetres $=Qn^2r_2{}^3$.
r_1=inner radius (centimetres). n=turns per centimetre.
r_2=outer radius (centimetres).

7. *Flat Coil of Best Time Constant.*

The (direct current) resistance of a flat coil wound closely with circular wire is

$$4\rho n^3(r_2^2-r_1^2)=4\rho n^3r_2^2(1-\alpha^2),$$

ρ being the resistivity of the wire, so that the time constant is proportional to $Qr_2/n(1-\alpha^2)$.

Also for a given length and section of wire $r_2^2(1-\alpha^2)$ and n are fixed, and therefore (by eliminating r_2) the time constant is proportional to $Q/(1-\alpha^2)^{3/2}$. By calculating this quantity for various radius ratios we obtain the variation of time-constant. It is found (*see* Table II.) that the maximum time constant occurs when the inner radius is about 2/5 the outer radius. The maximum is, however, very flat, the coil having 90 per cent. of the maximum efficiency if $r_1/r_2<0\cdot7$.

TABLE II.

$\alpha=0\cdot0$	0·1	0·2	0·3	0·4	0·5
$Q/(1-\alpha^2)^{3/2}=6\cdot97$	7·03	7·15	7·25	7·32	7·30
$\alpha=0\cdot6$	0·7	0·8	0·9	1·0	
$Q/(1-\alpha^2)^{3/2}=7\cdot18$	6·94	6·46	5·50	0·00	

The best time constant for a single layer *cylindrical* coil is obtained when the length (b) is 4/5 of the radius (a), and in that case $L=14\cdot90n^2a^3$, n again being the number of times per centimetre. For a flat coil of the best time constant

$$L'=5\cdot632n^2r_2^3.$$

To compare the two cases we require the radius a of the cylindrical coil which can be wound with the same length of wire as the flat coil. This is $a=0\cdot725r_2$, with $r_1=0\cdot4r_2$.

Using this $$L=5\cdot666n^2r_2^3.$$

The cylindrical coil is thus slightly better than the corresponding flat coil.

8. *Mutual Inductance between Coaxial Coplanar Flat Coils.*

The mutual inductance between two coaxial flat coils in the same plane can be obtained from Table I. as follows :—

Let the inner and outer radii of the $\left\{\begin{matrix}\text{inner}\\ \text{outer}\end{matrix}\right\}$ coils be

$$\left\{\begin{matrix}r_1, r_2\\ r_3, r_4\end{matrix}\right\}.$$

Denote the self-inductances of the two coils by L_α, L_γ, and that of the coil which would fill the interspace by L_β.

Denote the mutual inductances between these three coils by $M_{\alpha\beta}$, $M_{\beta\gamma}$, $M_{\gamma\alpha}$. Further, let L_α, L_β in series have self-inductance L_A; L_β, L_γ in series have inductance L_B; L_α, L_β, L_γ in series have inductance L_C.

Then $L_A=L_\alpha+L_\beta+2M_{\alpha\beta}$, $L_B=L_\beta+L_\gamma+2M_{\beta\gamma}$, $L_C=L_\alpha+L_\beta+L_\gamma+2M_{\alpha\beta}+2M_{\beta\gamma}+2M_{\gamma\alpha}$, from which the three M's can be found since the L's are known from Table I.

In particular

$$M_{\alpha\gamma}=\tfrac{1}{2}(L_C-L_A-L_B-L_\beta),$$

or, in terms of Q

$$M_{\alpha\gamma}=\tfrac{1}{2}n^2\{r_4{}^3(Q_{r_1/r_4}-Q_{r_2/r_4})-r_2{}^3(Q_{r_1/r_3}+Q_{r_2/r_3})\}.$$

A simple case is where there is no interspace, and the inner coil has zero inner radius. Then

$$M=\tfrac{1}{2}n^2R^3\{Q_0(1-a^3)-Q_a\},$$

in which $a=r/R$, R is the outer radius, r the dividing radius.

In order to give some idea of the magnitude of the various inductances, the following table has been calculated from the above formula :—

TABLE III.—*Mutual-inductances Between Flat Coils.*

α.	L_1/n^2R^3.	L_2/n^2R^3.	M/n^2R^3.	k.
0·1	0·00697	6·93	0·0162	0·074
0·2	0·0557	6·73	0·0930	0·152
0·3	0·1880	6·30	0·240	0·220
0·4	0·446	5·63	0·446	0·280
0·5	0·871	4·74	0·678	0·333
0·6	1·503	3·68	0·892	0·379
0·7	2·39	2·53	1·025	0·421
0·8	3·56	1·397	1·005	0·451
0·9	5·08	0·457	0·715	0·470
0·92	5·23	0·317	0·611	0·474
0·94	5·79	0·1910	0·495	0·470
0·96	6·17	0·0942	0·356	0·465
0·98	6·56	0·0272	0·192	0·454
1·00	6·97	0·0000	0·000	0·000

L_1=Self-inductance of inner coil.
L_2=Self-inductance of outer coil.
M=Mutual inductance between coils.
k=Coefficient of coupling=$M/\sqrt{L_1L_2}$.
r=Dividing radius.
$\alpha=r/R$. n=Turns per centimetre.

ABSTRACT.

Two formulæ are established for the computation of the self-inductance of single layer flat coils, one for the case when the inner and outer radii are not very different and the other for the case of small inner radius. The two formulæ are shown to be consistent and capable

of including all possible cases. From the formula a table is calculated which enables the inductance to be expressed in the form $L=Qn^2r^3$, in which n is the number of turns per cm., r the outer radius and Q is a tabulated function of the ratio of the inner and outer radii. Some applications of the table are given.

DISCUSSION.

Dr. **Eccles** recalled the very useful Paper published by the author some years ago, giving formulæ for thick coils. He had found these extremely useful, and felt sure that the results of the present Paper would prove equally valuable. He felt that people hardly realised their indebtedness to those who undertook these very laborious calculations for the help of the users of coils.

Dr. E. H. **Rayner** pointed out how useful such formulæ were in the construction of inductances. Almost anyone could make up a resistance to have any desired value; but it was another matter with regard to inductances. With formulæ such as these, however, it was possible with the simplest appliances to construct an inductance to within 1 per cent. of the required value.

Prof. G. W. O. **Howe** communicated the following: In the "Archiv für Elektroteknik," Vol. 3, 1915, page 187, is a Paper by J. Spielrein, entitled "The Self-Induction of Air-Core Spirally-wound Coils" (*see* "Science Abstracts," Elec. Eng., 1915, No. 660). Spielrein's final formula is $L=T^2r_2f(r_1/r_2)$, where T is the total number of turns, whereas Mr. Butterworth's formula is $L=n^2r^3Q$, where n is the turns per cm. Since $T=n(r_2-r_1)$ Spielrein's formula can be written $L=n^2(r_2-r_1)^2r_2f(r_1/r_2)$. Hence $Q=(r_2-r_1/r_2)^2f(r_1/r_2)=(1-r_1/r_2)^2f(r_1/r_2)$.

Now Spielrein gives a table of $f(r_1/r_2)$ to seven places for all values of r_1/r_2 from 0 to 1 by steps of 0·01. Comparing Spielrein's $f(r_1/r_2)$ with Mr. Butterworth's Q we have:—

r_1/r_0.	$f(r_1/r_2)$.	$(1-r_1/r_2)^2f(r_1/r_2)$.	Q.
0·0	6·969573	6·969573	6·970
0·5	18·97400	4·74350	4·743
0·9	45·74241	0·4574241	0·4574
1·0	∞	$0\times\infty$	0

It seems obvious that Mr. Butterworth was unaware of this Paper.

The Author, in reply, said he had not known that Spielrein had tackled the problem. He was afraid it rendered his Paper somewhat unnecessary.

Dr. **Eccles** pointed out that it was of great importance that two independent investigators obtained results in such striking agreement.

V. *An Experimental Method of Determining the Primary Current at Break in a Magneto.*

By N. W. McLachlan, *D.Sc., M.I.E.E.* (*From the National Physical Laboratory.*)

(Communicated by F. E. Smith, O.B.E., F.R.S.)
Received October 3, 1919.

In carrying out research on various types of magneto, it is sometimes necessary to know the value of the current broken in the primary circuit. Since the effective resistance and inductance of the primary winding under conditions prior to break are small, it is difficult to ensure that the insertion of apparatus in the primary circuit will not alter the usual working conditions. For this reason the oscillograph is out of the question for accurate measurement. Mr. A. P. Young* has devised graphical and analytical methods of determining the shape and magnitude of the primary current up to the point of break. With a view to facilitating the measurement of the primary current at break the following method has been developed.

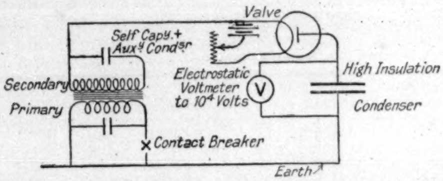

Fig. 1.—Diagram showing Arrangement of Apparatus for Reducing and Measuring the Peak Voltage of a Magneto.

Apart from the voltage after break due to rotation, the secondary peak potential of a magneto is very nearly proportional to the current broken in the primary circuit. Thus in order to determine the current broken at various speeds, the peak voltage is found and a curve plotted showing the relation between voltage and speed for any definite position of the timing lever. This curve must be corrected, the ordinates being reduced by an amount equal to the voltages due to rotation at the speeds in question. The apparatus used is shown in Fig. 1†. The voltage rises with the speed until in the neighbourhood of 100 revs. per min., sparks occur at the safety gap. In order to reduce the voltage of the magneto below that necessary to spark the safety gap, either a definite non-inductive leak can be allowed in the secondary circuit, or a condenser can be connected across the secondary terminals, or from the high tension lead of the secondary to earth. The condenser is probably preferable, since for a given reduction in peak voltage it will not affect the primary current before break as much as a resistance. The capacity (and the power factor) of the condenser should be as small as possible, and it should not leak. The capacity used in these experiments, including the distributed capacity of the secondary winding, was 600 picofarads. In this way a curve showing the relation between secondary peak voltage and speed can be obtained. In order to determine the voltage after break due to rotation, the magneto is run at various speeds with the condenser on the secondary, the contacts of the breaker being insulated, *i.e.*, on open circuit. The relation between voltage and speed with the condenser was not quite linear, owing to the increased condenser effect at high speeds. The effect of the condenser was to increase the voltage due to rotation by 15 per cent. at 1,200 revs. per min. The voltage determined in this case was the rotational peak voltage. To estimate the fraction of the latter voltage, which must be subtracted from the total peak voltage in order to obtain that due solely to interruption of the current, the open circuit voltage curve is obtained by means of an oscillograph, or a point to point method. In using the latter method, it is necessary to find the rotational open circuit voltages at advance and retard

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* "Journal" Aeronautical Society, No. 82, April, 1917.

† Campbell and Paterson, "Phil. Mag.," Vol. XXXVII., March, 1919.

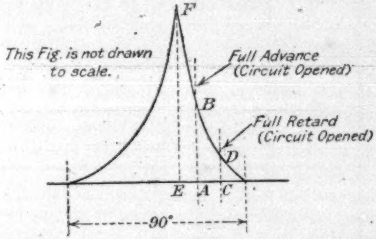
F
This Fig. is not drawn to scale.
Full Advance (Circuit Opened)
B
Full Retard (Circuit Opened)
D
E
A
C
90°

Fig. 2.—Diagram showing Primary Open Circuit Voltage Wave of Magneto.

only. When the open circuit voltage curve is used, a vertical line is drawn at either the full advance or full retard positions, as shown in Fig. 2. This curve is taken from Mr. Young's paper (*loc. cit.*), and may be adopted for the magneto used in these experiments without introducing appreciable errors.* The open circuit peak voltage due to rotation is *EF* and the voltage at advance break is *AB* and at retard *DC*. Knowing the ratios *AB*/*EF*, *DC*/*EF*, and the value of *EF* as found from a speed calibration test, the value of the voltage to be subtracted from the total peak voltage can be calculated. Allowance can be made for the lag of the peak voltage behind the breaking of the current, if necessary, by shifting *AB* and *CD* to the right by the requisite amount, but this refinement is probably superfluous.

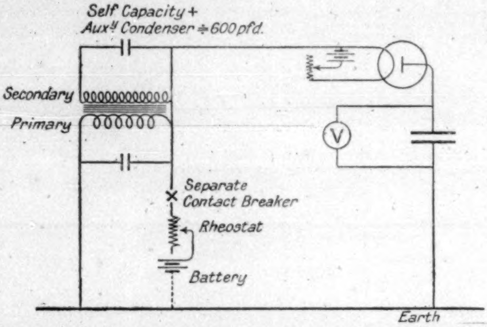

Fig. 3.—Arrangement of Circuit used for Calibrating Magneto.

The system is then calibrated by using the circuit shown in Fig. 3.

A battery and non-inductive rheostat are connected in circuit with a separate contact breaker and the primary of the magneto. In order to reduce earth capacity effects (*see* Fig. 6), and the effect of transients through the battery and rheostat before and after break to a minimum, the connections of Fig. 3 should be adopted. In this respect it is advisable to use a portable battery in preference to a portion of a large main battery. The armature of the magnets is clamped in the position at which break would occur during rotation, *e.g.*, full advance or full retard. Precautions are taken to ensure a sufficiently small time constant, so that the current through the armature attains its maximum value before the circuit is broken. The current should produce a magnetic field *in the same direction* as that of the induced current when the magneto is operated in the usual manner. In order to allow for leakage from the peak voltage measuring apparatus, the contact breaker should run at about 600 revs. per min. With two breaks per revolution this gives 20 breaks per second. A series of readings of current broken in the primary of the magneto and the peak voltage are taken.

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* On full advance the correction is about 15 per cent. when the speed is 2,000 revs. per min., so that an error of 10 per cent. in the voltage due to rotation is only an error of 1·5 per cent. in the primary current at break. On retard the correction is 2·5 per cent. at 2,000 revs. per min. The correction decreases with decrease in speed, and is very small on the initial portions of the curves of Fig. 4.

There is a slight difference between the working conditions and those during calibration. Under running conditions, using a condenser across the secondary, the current through the armature and the flux are reversed, whereas with a fixed rotor and a separate contact breaker the flux is never reversed. The induced voltage wave of the magneto is peaked, whereas with the contact breaker and a battery the voltage is constant. Moreover, the flux change in the iron before break is different in the two cases. Thus the magnetic history of the iron will be different. These variations in conditions, however, ought not to have a very appreciable effect, and it is probable that the results are sufficiently accurate for all practical requirements. Whatever may be the difference between the two conditions, the method yields results which are useful for purposes of comparison.

In order to ascertain the effect of the condenser across the secondary on the value of the current broken in the primary, found by the method already described, comparative tests were carried out with and without the condenser. The voltmeter employed would not read above 10,000 volts, owing to the nearness of the fixed and moveable vanes, so that readings could not be taken at a speed of more than 100 .evs. per min. This difficulty was overcome by using either a core of thick iron, *e.g.*, strips about 4 mm. thick, or a core of the same type as that used in practice, but of half the cross-sectional area. The actual core was also used, the voltage being reduced by introducing air gaps into the magnetic circuit between the laminated vertical limbs in the housing and the core. The peak voltage was found at various speeds up to 2,000 revs. per min. with and without the condenser for different positions of the timing lever. The ratio of the peak voltages, with and without the condenser at full retard, was about 3 per cent. less at 80 revs. per min. than at 2,000 revs. per min. On full advance there was a closer agreement at all speeds.

The ratio of the peak voltages found by direct current calibration, was about 2 per cent. less than that at 80 revs. per min. and 5 per cent. less than that at 2,000 revs. per min., when the timing lever was fully retarded. At advance the agreement was better. Thus, considering the difficulty of obtaining accurate results in magneto research, the effect of the condenser, although it increases slightly with the speed, is not of much importance. The increase in peak voltage is probably chiefly due to the effect of the condenser on the behaviour of the iron (it decreases the damping 27 per cent. owing probably to the reduction in the frequency of the oscillations), but there is doubtless a slight increase in the primary current at high speeds. It was not always possible to repeat the readings to within 1 per cent. Variations of 3 per cent. were obtained and it was observed that these were connected with the previous history of the iron.

The current at break can also be found by using an armature having the same type of core and the same number of primary turns as that used with the condenser. The number of secondary turns must be reduced from 10,000 to 3,000, so that the peak voltage can be obtained at all speeds. The armature is calibrated in the manner indicated above. Using an armature of this type there was a close agreement between the values obtained by the two methods at full advance, but an increase of 20 per cent. at low speeds with the timing lever fully retarded. At high speeds the armature gave values about 2 per cent. low. These differences can probably be accounted for by inequality of the iron cores, and by the fact that the small armature was wound on a brass former whilst the other was not. In the advance and retard positions the configuration of the magnetic circuit and the polarisation due to the magnet are different, owing to the altered position of the rotor.* The value of dB/dt is different, and the magnitude of the negative flux is greater in the latter case than in the former. The highest flux density in the armature is greater at advance than at retard, but the total flux change between make and break is greater at retard than at advance (*see* Fig. 9). It is for this reason that the current and energy at high speeds, with the timing lever fully retarded, are greater than the same quantities at full advance. At low speeds the current and energy at break are less on retard than advance, because in the one case break occurs after the current has attained its maximum value, whereas in the other it occurs when the current is almost a maximum. This is due to the fact that at low speeds the effective resistance of the primary winding is of greater importance than the inductive reactance. Thus as the speed increases, the angle (on the time axis) between make and the occurrence of the maximum primary current also increases. Owing to eddy current and hysteresis losses the effective resistance of the primary increases with the speed. Since the screening effect of the eddy currents increases with the speed, the effective permeability of the iron decreases, and therefore the inductance of the primary in any given position decreases. Hence, there is a certain speed for which the current at break is a maximum.

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* The rotor was partially laminated, and the proportion of the oscillatory magnetic circuit which was laminated at retard, was greater than at advance.

It follows from the preceding that two apparently similar cores which gave approximately the same value of the current at advance would not necessarily do so at retard. Another armature having standard primary and secondary windings, viz., 150 turns of 22 S.W.G. and 10,000 of 41 S.W.G. enamelled wire, wound on a brass former, was tried. The readings were in close agreement at advance, but at retard they were 15 and 4 per cent. less at 90 revs. per min. and 1,700 revs. per min. respectively, than those obtained with the reduced armature. At low speeds on full retard or advance it was found that the primary current was the same with and without the condenser. It is evident, therefore, that the performance in these cases depends chiefly on the quality of the iron cores, although with a condenser across the secondary the transient oscillations which occur after make, may influence the magnetic behaviour of the iron. It is desirable in practice to use a condenser across the secondary in preference to an armature with a reduced secondary. In the former case the current under actual working conditions is obtained and assumptions have not to be made with regard to the quality of the iron cores.

The results of some experiments on an inductor type, B.T.H., A.V.8s magneto* are given in Fig. 4. It should be mentioned that the magnet was not magnetised to its maximum extent, since the machine had been dismantled before the tests were carried out. This, however, only affects the absolute magnitude of the current in the armature.

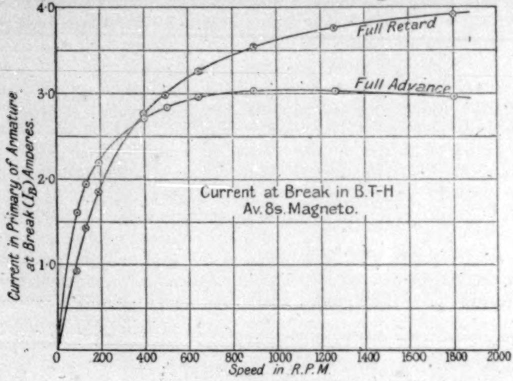

FIG. 4.

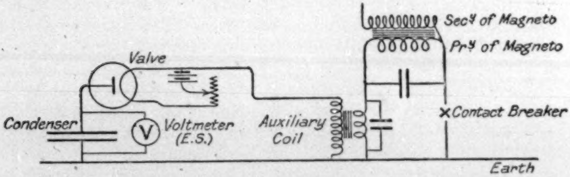
Sec^y of Magneto
Pr^y of Magneto
Valve
×Contact Breaker
Condenser
Voltmeter (E.S.)
Auxiliary Coil
Earth

Fig. 5.—DIAGRAM OF APPARATUS FOR MEASURING THE PRIMARY CURRENT AT BREAK IN A MAGNETO BY THE AID OF AN AUXILIARY INDUCTION COIL.

It may be of interest to outline another method of measurement which was tried, but without success. The circuit used is illustrated in Fig. 5.

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* The experiments described herein refer to this type of magneto, unless otherwise stated.

An auxiliary iron-cored induction coil with a primary of very small inductance (5 microhenries at 50 ∿) and resistance 0·005 ohm, is inserted between one end of the primary winding of the magneto and earth (not a virtual but an actual earth). There is no mutual inductance between the coil and the magneto. The auxiliary coil, as shown in Fig. 5, does not appear to be in the primary or secondary oscillatory circuits after break, provided the magneto is *prevented from sparking*, and provided the high-frequency oscillations in the primary do not pass through the initial arc or spark when the contacts of the breaker separate.

A condenser of about 0·1 microfarad is connected across the terminals of the auxiliary coil to prevent sparking at the contact breaker, since the magneto condenser is not connected directly across the breaker. The secondary winding of the auxiliary coil is connected to earth and to the filament of a two-electrode ionic valve which serves to rectify the oscillations. The anode of the valve is connected to an electrostatic voltmeter reading to 160 volts with a condenser in parallel to allow for voltmeter leakage, which latter must be extremely small. The other terminal of the voltmeter is connected to earth. It would appear that the reading of the voltmeter is a measure of the current broken in the primary of the auxiliary coil, and therefore of the current in the primary of the magneto. This, however, was found not to be the case as will be explained later.

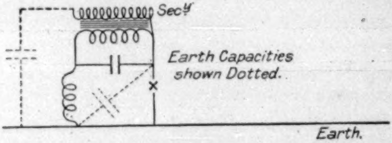

Fig. 6

It is well known that sparking or flashing occurs periodically at the contact breaker of a magneto. It was found that the voltmeter was subjected to violent impulses from time to time, the intervals between the impulses decreasing with increase in the speed of rotation. They were particularly noticeable during changes of speed and at high speeds. At low speeds their effect could be eventually wiped out by allowing a slight leak on the voltmeter, and by increasing the primary capacity. This, however, caused a large error in the reading. It was suspected that these impulses were due to the oscillations in the primary of the magneto traversing the primary of the auxiliary coil via the arc at the contact breaker. There is also another possible reason for the passage of oscillatory current through the primary of the auxiliary coil. This is due to capacity effect to earth of the magneto windings. This can be represented by condensers as shown dotted in Fig. 6.

If the magneto is allowed to spark, high frequency oscillations pass through the auxiliary coil and increase the reading of the voltmeter enormously.

The auxiliary coil is calibrated using the circuit of Fig. 3. Owing to the voltmeter leakage being large to allow for impulse effects, the reading increased with the speed of rotation of the contact breaker. The method is, therefore, unreliable.

Investigation into the Effect of a Condenser, Across the Secondary Terminals of a Magneto on the Primary Current at Break.

The induced voltage wave of a magneto is peaked as shown in Fig. 2, and therefore contains pronounced higher harmonics. In the analytical treatment of the above problem we will for simplicity, since the condenser effect is only of the second order of magnitude, assume that the induced primary and secondary voltages are sinusoidal in wave form. It is the slope of the voltage wave which is the determining factor for the capacity current, but we can virtually increase the slope by taking an equivalent frequency greater than that of the alternation of the flux. Thus, at high speeds of, say, 1,800 revs. per min. the effect of capacity will be much greater than at a speed of 80 revs. per min.

The oscillations which occur after interruption of the primary current are assumed to have died away in a time which is small compared with the interval between break and make. In experiments on the same magneto as used previously, it was found that the ratio of the first maximum of the voltage wave to the first minimum was 1·8 with the condenser and 2·5 without it. Thus, assuming a mean frequency of 5,000 ∿ per second, the oscillations would have been reduced to 1 per cent. of their original value after 8×10^{-4} sec. with and 6×10^{-4} sec. without the condenser. At 1,800 revs. per min. the time between break and make is about 5×10^{-3} sec., so that the above assumption is justifiable. These oscillations will have an effect on the magnetic behaviour of the iron.

Since the induced current and voltage waves in the magneto, *before break*, are not sinusoidal, and since the permeability of the iron is not constant, the values of the primary and secondary inductances and effective resistances cannot be considered constant. Owing to the fact that the condenser effect is small, it is sufficiently accurate to assume that the inductances and effective resistances are constant and have the same values as those measured at the equivalent pulsatance ω, in this case about $1{,}500 \div 250 \sim$ per second, or four times the frequency of the flux.

Between break and make there is a condenser in both the primary and secondary circuits. The capacity reactance is very large compared with the resistance, and the latter can therefore be neglected. After make, there is no capacity in the primary circuit, but the effective resistance is comparable with the inductive reactance, so that it cannot be neglected.

Let L_1 L_2=effective inductance of primary and secondary windings, assumed constant;

M=mutual inductance, assumed constant, also that $L_{21}=L_{12}$;

k=coefficient of coupling;

q_1 and q_2=quantities of electricity in primary and secondary condensers;

i_1 and i_2=currents in primary and secondary circuits;

C_1 C_2=capacities in primary and secondary circuits;

R_1=effective resistance of primary winding, assumed constant;

$\frac{E_1}{E_2} \doteq \frac{n_1}{n_2}$=ratio of turns on primary and secondary windings.

Considering the rotational effect between break and make, after the main high frequency oscillations due to the cessation of the primary current have died away, we have for the primary circuit

$$L_1 D^2 q_1 + M D^2 q_2 + \frac{q_1}{C_1} = E_1 \sin \omega t \quad . \quad . \quad . \quad (1)$$

where $e_1 = E_1 \sin \omega t$ is the E.M.F. induced in the primary due to rotation and $D = d/dt$.

For the secondary circuit the equation is similar

$$L_2D^2q_2+MD^2q_1+\frac{q_2}{C_2}=E_2 \sin \omega t. \quad . \quad . \quad . \quad (2)$$

The arrangement of the circuits is shown in Fig. 7.*

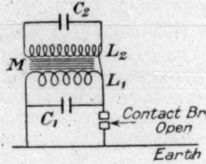

Fig. 7.

Solving these equations we obtain the primary current,

$$i_1=\frac{\frac{E_1}{\omega L_1}\left[\frac{\omega^2}{L_2C_2}+\omega^4\left(\frac{M}{L_2}\cdot\frac{n_2}{n_1}-1\right)\right]\cos \omega t}{\left(1-\frac{M^2}{L_1L_2}\right)\omega^4-\left(\frac{1}{L_1C_1}+\frac{1}{L_2C_2}\right)\omega^2+\frac{1}{L_1L_2C_1C_2}}+$$

a transient which dies away rapidly, and can therefore be neglected . . . . (3)

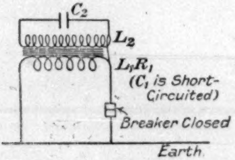

Fig. 8.

When the contacts close equation (2) remains unaltered, but (1) becomes (*see* Fig. 8)

$$L_1D^2q_1+MD^2q_2+R_1Dq_1=E_1 \sin \omega t \quad . \quad . \quad . \quad (4)$$

Solving (2) and (4) we obtain

$$i_1=\text{const.}\times e^{-\frac{R_1}{L_1}t}+\frac{F\omega[G\cos \omega t+\omega H \sin \omega t]}{H^2+\omega^2G^2}+$$

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* The analysis has been worked out on the assumption that the primary and secondary circuits are coupled, but not directly connected, as shown in Fig. 7. This does not introduce appreciable errors in (7).

a transient which can be neglected, since it will have died away long before break . . . . (5);
where

$$F=\frac{E_1}{L_1}\left[\frac{1}{L_2C_2}+\omega^2\left\{k\cdot\frac{n_2}{n_1}\left(\frac{L_1}{L_2}\right)^{\frac{1}{2}}-1\right\}\right];$$

$$G=\omega^2\left[\omega^2(1-k^2)-\frac{1}{L_2C_2}\right];$$

$$H=\frac{R_1}{L_1}\left(\frac{1}{L_2C_2}-\omega^2\right);$$

$$\text{Const}=\frac{-F\omega G}{G^2+\omega^2H^2};$$

since $i_1 \doteq 0$, when $\omega t=0$, for its value from (3) is very small.

If $C_2=0$, *i.e.*, there is no secondary condenser, and $k=0$, we obtain from (5)

$$i_1=\frac{E_1}{(R_1{}^2+\omega^2L_1{}^2)^{\frac{1}{2}}}[e^{-\frac{R_1}{L_1}t}\sin\theta+\sin(\omega t-\theta)], \quad . \quad (6)$$

where $\theta=\tan^{-1}\omega L_1/R_1$; as we should expect, since the primary has then only resistance and inductance, and the secondary is inoperative.

Thus, since the self-capacity of the secondary is small, the effect of the condenser on the primary current is obtained by subtracting (6) from (5).

The following numerical values for the constants of the primary and secondary circuits will be taken :—

$L_1=5\times10^{-3}$ henry,
$L_2=25$ henries,
$C_2=6\times10^{-10}$ farad (=self capacity of secondary+condenser)
$R_1=2$ ohms,
$\omega=1{,}500$ or $f=250$ ∾ per sec.,
$n_2/n_1=65$,
$k^2=0{\cdot}9$.

Substituting these values in the expressions for F, G and H we find that the quantities

$$\omega^2\left[k\frac{n_2}{n_1}\left(\frac{L_1}{L_2}\right)-1\right];\ \omega^2(1-k^2);\ \omega^2,$$

can be neglected. Thus we can write

$$F=\frac{E_1}{L_1L_2C_2},\quad G=\frac{-\omega^2}{L_2C_2}\quad\text{and}\quad H=\frac{R_1}{L_1L_2C_2}.$$

Hence, substituting these values in (6) we obtain,

$$i_1\doteqdot\frac{E_1}{(R_1^2+\omega^2L_1^2)^{\frac{1}{2}}}\left[e^{-\frac{R_1}{L_1}t}\sin\theta+\sin(\omega t-\theta)\right].\quad .\quad (7)$$

Since (6) and (7) are very nearly equal, it follows that a condenser of 600 picofarads across the secondary circuit has no appreciable effect on the primary current at break. The same reasoning holds if $i_1=0$ when $\omega t=a$, and break takes place when $\omega t=a+\pi/2$. In either case the effect of the condenser is less than 1 per cent., assuming no dielectric loss.

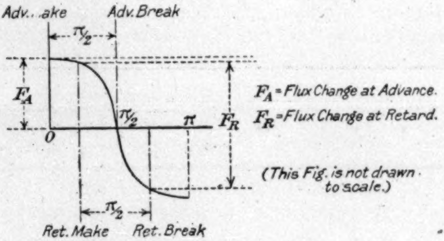
Adv. Make
Adv. Break
π/2
F_A
0
π/2
π
F_R
π/2
Ret. Make
Ret. Break
F_A = Flux Change at Advance.
F_R = Flux Change at Retard.
(This Fig. is not drawn to scale.)

Fig. 9.—Diagram showing Flux Change in Magneto at Full Advance and at Full Retard.

The diagram is only an approximate representation of the flux changes at advance and retard. The actual slope at $\pi/2$ is greater than that shown, and when advance break occurs the flux is negative.

The relative displacements of make and break on the time axis at full advance and full retard for a B.T.-H. inductor type magneto are shown in Fig. 9.

Since the foregoing was written some experiments have been carried out with a Thomson-Bennett E.4 rotating armature magneto. In this machine the peak voltage and, therefore, the primary current, is greater at all speeds on advance than on retard. This is due partly to the design of the cam for operating the contact breaker, since it controls the relative positions of make and break, and also to the setting of the timing lever. The capacities of the condensers necessary to reduce the peak voltage below that to cause sparking at the safety gap at 1,800 revs. per min., were 1,800 picofarads on full retard and 3,800 picofarads on full advance. Both of these values are greatly in excess of that used with the B.T.-H. machine.* The influence of the condenser on the primary current would be greater at advance than at retard, its magnitude being from 3 to 4 per cent. at 1,800 revs. per min. With the B.T.-H. inductor type magneto a condenser of 3,800 picofarads would have a greater effect, since there are two alternations of flux per revolution, whereas in the Thomson-Bennett there is only one flux alternation per revolution.

With the Thomson-Bennett machine the open circuit voltage was greater than with the B.T.-H. On full advance, where the rotational effect is greatest—assuming the same phase relations and wave form as those of Fig. 2†—the voltage due to rotation is 20 per cent. of the total peak voltage at 2,000 revs. per min., if the latter is controlled by the safety gap. A correction of this magnitude is too large. In general it is not of great importance to know the value of the primary current at speeds above 1,000 revs. per min. If, however, it is necessary to exceed this speed of rotation, some method of reducing the proportionate effect of the rotational voltage must be adopted. This can be accomplished in two ways : (*a*) using an armature with a reduced secondary in which case no condenser is required ; (*b*) cutting the safety gap out of action, using an electrostatic voltmeter reading to 15,000 volts‡ and reducing the size of the auxiliary condenser. In either of these cases the machine has to be partially dismantled, which is rather a laborious process if method (*a*) is adopted.

ABSTRACT.

A method of obtaining experimentally the current at break in a magneto is described. A condenser is connected across the secondary winding to reduce the voltage below that to cause sparking at the safety gap. The peak voltage due solely to interruption of the current at any speed is found. The interrupted direct current necessary to give the same peak voltage is also found by using a calibrating circuit.

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* As indicated above the magnet of this machine was not fully magnetised. The designs of the two types of machine, however, are quite different.

† It is advisable, of course, to determine the secondary open circuit voltages at retard and advance for the particular machine under test.

‡ This is about the voltage at which a rotating armature magneto sparks across at the slip-ring when the safety gap is removed.

The magnitude of this current is equal to that broken in the magneto. The influence of the secondary condenser on the primary current at high speeds is discussed.

DISCUSSION.

Dr. **Norman Campbell** (communicated the following): It is quite possible to make satisfactory measurements on the primary current of a magneto by inserting in the circuit a small non-inductive resistance and taking the potential difference across it by means of a rotating "contactor." The total resistance inserted need only be 0·03 ohm; 0·01 for the leads and 0·02 for the measuring resistance. By the use of a suitable potentiometer method a change of 0·02 amp. can then be detected, which is ample for practical purposes. It was found that on increasing the total resistance inserted to 0·11 ohm, the current did not change by as much as 3 per cent.; accordingly the disturbance caused by the resistance of 0·03 ohm must be quite inappreciable.

With this arrangement it is possible to test the accuracy of Dr. McLachlan's method by examining whether the secondary peak potential is proportional to the primary current. It was found that it was very accurately proportional when the current was changed by varying the strength of the magnets. By changing that strength within wide limits it seems possible to change all currents and potentials in the circuits without changing appreciably their ratio or wave form.

On the other hand, considerable discrepancies, amounting sometimes to 10 per cent., were found when the secondary peak potential produced by breaking a primary current generated by rotation of the armature was compared with that produced by breaking the same current supplied by a battery. My experience indicates that errors of this order might occur in using Dr. McLachlan's method (allowance, of course, was made for the E.M.F. generated directly in the secondary by rotation of the armature).

For the rest, my experiments agreed in general with those of Dr. McLachlan. Since I have not kept the records I cannot discuss them in detail; but I remember being surprised how different was the open-circuit voltage wave which I found from that shown in Fig. 2.

Dr. **Eccles** said that Dr. McLachlan had used an indirect method and then made a series of check experiments to show that the various errors introduced are negligible. Dr. Campbell had mentioned a direct method. He thought that direct methods were preferable where possible; and suggested that the Dufour cathode ray oscillograph would have been the ideal instrument for the investigation of the magneto.

Dr. **Rayner** said there was one point about which he felt doubtful in the direct current calibration. The calibrating flux was not employed in the same way as the working flux, one being reversed and the other not. He thought some method might be devised to avoid this discrepancy. He was afraid the power required to work the Dufour oscillograph might be greater than was readily obtained from a magneto.

Dr. **Eccles**: An amplifier could be used.

Prof. **Lees** pointed out that the important thing in these researches was that we were obtaining knowledge about the magneto, which has hitherto been treated almost wholly on empirical lines.

The **Author**, in reply, stated that he had tried the method outlined by Dr. Campbell, but without the use of a rotating contactor. The results were unsatisfactory owing to the effect of transients before and after break. In order to eliminate these, it is essential that the contactor circuit should be broken before the primary. This, however, does not eliminate transients *before* break. The effect of these will be evident from the remarks in the Paper on the auxiliary coil method. Unless the interval between the breaking of the two circuits is small, errors can easily arise. Dr. Campbell's statement regarding resistance only applies at high speeds [illegible], say, 700 revs. per min. or more. At low speeds, owing to the small value of the primary inductive reactance, the current is nearly proportional to the resistance. This will be seen from Fig. 4, since the current-speed relation is linear, and the E.M.F. is proportional to the speed. Hence, increasing the resistance to 0·03 or 0·11 ohm, as Dr. Campbell did, would entail a reduction in current of 6 and 22 per cent. respectively on the initial portions of the curves of Fig. 4.

If the method of measuring the voltage drop across the resistance is one which gives the peak value, the current at break is measured only when the maximum value does not occur *before* break. If the maximum occurs before break, as it does at retard, the peak voltage across the resistance is not a measure of the current at break, although the values so obtained may be proportional to the secondary voltage of the magneto found by altering the strength of the magnet. The question arises as to whether the peak voltage—due solely to interruption, as found by Dr. Campbell's method—is proportional to the current broken when the *speed* is varied.

There are several ways in which discrepancies may arise in measuring the peak voltage by breaking a direct current in a magneto. (1) If the direction of the current is different from that during rotation, errors of the order mentioned by Dr. Campbell may occur. This can be tested by moving the armature through 180 electrical degrees and repeating the experiments. (2) The position of the armature at break must be found fairly accurately, since the polarisation of the core due to the magnet varies with the armature position. Polarisation affects the primary and secondary effective resistances and conductances, also the flux change due to a given current. These react on the peak voltage. (3) The peak voltage found by breaking a direct current (flowing in the proper direction) at advance or retard is not the same for all armature positions. This is probably due to some asymmetry. (4) The rectifying valve should pass no reverse current, and at low armature speeds the leakage should be extremely small. (5) Great care must be taken to prevent sparking at the separate contact-breaker used for calibration. If sparking, or rather arcing, occurs, there are variations in the peak voltage, and the battery and rheostat form part of the oscillatory circuit.

Unfortunately, Dr. Campbell does not state (*a*) the type of magneto, (*b*) the values of the currents, (*c*) the speeds, (*d*) the position of the timing lever, (*e*) whether the 10 per cent. error was in excess or defect. In practice one is generally concerned with currents of 2 amperes or more, and there does not seem to be any definite reason why the method given in the Paper should be subject to errors of 10 per cent. in measuring currents of this magnitude if the precautions outlined above are taken.

Dr. Eccles' suggestion to use the Dufour oscillograph for measurements on the magneto was contemplated eight months ago. Up to the present, however, it has not been possible to secure one of these instruments. Without oscillograms of the voltage wave form obtained by apparatus of this nature it is impossible to know precisely what happens in the secondary of a magneto after break. With regard to Dr. Eccles' reference to the use of an amplifier to measure the primary current, difficulties may arise due to transients which would influence the valve circuits. Owing to the fact that the magneto radiates oscillations of very short wave length (10 to 30 metres), a further source of trouble with the amplifier might arise. This, however, is a matter for experiment.

The point raised by Dr. Rayner, and mentioned in the Paper, should not have serious consequences for currents of 2 amperes or more. In view of the fact that an appreciable variation exists between the performances of magnetos of the same type, it is not necessary to measure the current more accurately than 5 per cent. This will be clear when one remembers that only about 30 per cent. of the prmary electromagnetic energy in the form $\frac{1}{2}LI^2$ is transformed into electrostatic energy in the secondary. The chief object in the determination of the primary current was to obtain an estimate of the value of $\frac{1}{2}LI^2$.

VI. *Note on a Modified Form of Wehnelt Interrupter.* By F. H. Newman, *A.R.C.Sc.*, *B.Sc.*, *University College, Exeter.*

Received Oct. 7, 1919.

The Wehnelt interrupter is not used generally owing to the rapid disintegration of the platinum wire, and also because the current required for the working of the cell is large. If, however, a modified form is adopted these disadvantages disappear. A plate of aluminium, area 50 sq. cm., takes the place of the lead plate, the other electrode being platinum wire sealed in a glass tube. These electrodes are immersed in a saturated solution of ammonium phosphate made alkaline with ammonia. This interrupter works equally well with direct or alternating currents. In the former case the wire must be made the positive electrode, while with alternating currents the passage of the current is such that the wire is positive to the aluminium. The cell works with a minimum potential difference of 18 volts, but the frequency of the interruptions increases with the applied potential difference. The current produced is intermittent, and depends on the surface area of the wire, becoming greater as the surface area increases. Thus, with a wire 5 mm. long and 0·3 mm. diameter, the current with a P.D. of 30 volts is 1·0 ampere, which increases to 1·4 amperes, when the P.D. across the terminals is 100 volts. With a wire 20 mm. and 0·3 mm. diameter the current is 3 amperes for a P.D. of 35 volts. Working with a 6 in. induction coil the current in the secondary is very steady and unidirectional, and can be used for X-rays or where a steady unidirectional discharge is required for any length of time. The interrupter with 5 mm. of wire has worked continuously for 10 hours without much disintegration of the platinum wire or heating of the liquid. The ordinary Wehnelt interrupter could not be used for such a long period. It was placed directly across the mains of 100 volts alternating current, and connected in series with the primary of the induction coil. No condenser nor make-and-break is required, and the great advantage is the small current necessary for the working of the interrupter, and yet the secondary discharge is quite steady and of high tension. For the production of unidirectional secondary currents this interrupter works very well, and requires little attention.

DISCUSSION.

Capt. C. E. S. **Phillips** said it was important that the experiments should be repeated on the scale required for ordinary X-ray work. He would be glad to afford the author any facilities he required for this purpose. Had he tried the use of potassium carbonate instead of ammonium phosphate? In the X-ray service in the Army a reliable portable interrupter was badly required. The objection to the ordinary Wehnelt was that a voltage of 80 to 100 was required.

Mr. W. R. **Cooper** asked if rectification was essential to the working of the instrument. Would plates of other metals than aluminium work?

Mr. **Newman** said he had not tried other salts. Any metal would serve, but aluminium was best and required the least voltage.

DISCUSSION ON LUBRICATION

HELD AT

THE IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY

ON NOVEMBER 28TH, 1919.

PAPERS IN TYPE BEFORE THE MEETING.

Oiliness and Lubrication. *By* R. MOUNTFORD DEELEY.

COMMUNICATED BY THE LUBRICANTS AND LUBRICATION COMMITTEE.

WHEN metallic, or other surfaces, in contact are caused to slide over each other, considerable frictional resistance to motion results and the surfaces abrade each other. In machinery there are numerous rubbing surfaces, and the materials in contact must slide over each other with as little frictional resistance and as little wear as possible. Engineers have found by experience that the resistance to motion of such bearing surfaces, and the amount of abrasion or wear they suffer may be greatly reduced by interposing between them certain liquids or greases, known as lubricants.

Not all liquids will act as lubricants. Indeed, it has always been necessary to ascertain by actual trial what liquids are suitable and what are not. Until the growth of the heavy mineral oil industry, almost all the lubricating oils and fats used were either of vegetable or animal origin, and engineers found by experience which particular oil or fat was suitable for any particular machine. Given the required lubricant, the proper running of the machine was guaranteed. But with the enormous growth of our industries depending upon machinery for their existence, the demand for lubricants became so great that the supply of vegetable and animal oils was not sufficient to meet the requirements, and the mineral oils were pressed into service whenever possible.

To ensure the use of suitable lubricants, the chemist had to be consulted. Although he could not ascertain by any chemical or physical tests whether a particular oil or fat was a good lubricant, he could say with some certainty whether a sample was genuine or adulterated, and knowing the kind of adulterant used, he could give useful advice to the purchaser. This was especially the case with animal and vegetable oils, but mineral oils could not be so satisfactorily dealt with.

The reason why one liquid should be a good lubricant and another not, has attracted a good deal of notice by physicists, chemists and engineers. Attention was first directed to viscosity; for it was found that the viscosity of the fluid was a very important consideration in lubrication problems. The exact part played by viscosity was made apparent by the experimental work of Beauchamp Tower and the theoretical consideration of his results by Osborne Reynolds. It appeared that when a liquid was applied to two surfaces in contact which were in relative motion, and certain conditions as to speed, &c., were fulfilled, that the oil became "trapped" between the surfaces and actually entirely separated them. The friction was then proportional to the mean thickness of the oil film, the rate of shear and the viscosity of the oil. This was a great step in advance, for it was apparent that any viscous liquid might act as a lubricant if such conditions were adopted as would ensure a proper film being formed. Engineers have not been slow to profit by the discoveries of Tower and Reynolds, for every care is now taken in the design of machinery to secure, wherever possible, the presence of a proper lubricating film.

The formation of a proper lubricating pad or film cannot, however, always be relied upon. At very low speeds and under very heavy loads such a film will not form, and the solid surfaces rub against each other. In other cases the lubricant is only held in position by capillary forces, and a film of sufficient thickness does not form. Indeed, there are many cases in which it seems impossible to form such a film. The slide valves of steam engines, the pistons of various kinds of engines and the slow moving pulleys of cranes are cases in point. Efficient lubrication then depends upon the liquid possessing some other property which enables it to reduce the frictional resistance sufficiently. It thus came about that a true lubricating oil was regarded as a liquid possessing *oiliness.*

So far, it would appear that no satisfactory explanation has been given as to what oiliness really is. All that can be said is that some liquids of similar viscosity are not oily in this sense and will not act as lubricants.

Vegetable and animal lubricants are compounds of carbon, hydrogen and oxygen termed esters, formed by the union of alcohol radicles with fatty acid radicles. There are a great number of such fatty esters, several of which generally go to form each lubricant. The alcohol radicle occurring in the vegetable oils and in most of the animal oils is the trivalent radicle glyceryl C_3H_5 and its esters, known as glycerides, may be represented by the formula $C_3H_5R_3$, in which R represents the fatty acid radicle. Thus we have triglyceryl stearate, $C_3H_5(O.C_{18}H_{35}O)_3$; triglyceryl palmitate, $C_3H_5(O.C_{16}H_{71}O)_3$; triglyceryl oleate, $C_3H_5(O.C_{18}H_{33}O)_3$; and many others. Stearin and palmitin predominate in the solid fats, olein in the fluid oils. Olein is the chief constituent of many of the best known lubricants, such as lard, tallow, neatsfoot and olive oils. Allied to the fats are the waxes, which are esters of mono- and di-valent alcohols, and a few lubricants such as sperm oil belong to this group.

Mineral lubricating oils are of entirely different chemical composition, consisting essentially of hydrocarbons, compounds of carbon and hydrogen, and we labour under the disadvantage of knowing little concerning their ultimate chemical structure. Dunstan and Thole,* however, have considered the peculiarities of these oils and conclude that all good lubricating mineral oils contain unsaturated molecules. They remark: "In no case has the chemical constitution of a component of a lubricating oil been established, but the chemical behaviour of these oils indicate that among the components are unsaturated hydrocarbons (possibly open-chain, but more probably naphthenic and polynuclear, or perhaps of both types), saturated hydrocarbons (naphthenic and probably to some extent polynuclear, but not to any appreciable extent paraffinoid), and aromatic hydrocarbons (to an unknown and possibly a limited extent) . . . . The unsaturated compounds constitute between 20 and 40 per cent. of most lubricating oils . . . . It appears, then, that the true lubricant is an unsaturated compound, possessing all the attributes of such a compound, *i.e.* :—

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* "Journal" of the Institution of Petroleum Technologists, Vol. IV. (1918), pp. 191-228.

(1) Capacity to absorb iodine, bromine, oxygen, and so on ;
(2) Solubility in strong sulphuric acid ;
(3) Higher C/H ratio than the saturated derivative.

"Apparently the same facts hold good in regard to fatty lubricants. Rape oil, castor oil and olive oil contain in their molecules double bonds, and are superior in body and viscosity to such a saturated product as, for example, tallow.

"In recent years the progress of organic chemistry has largely been due to the realisation that unsaturation, or the possession of residual affinity, plays an all-important part in the reactivity and the very personality of a compound. Colour, odour, taste, physiological activity, and, in a word, all the characteristic properties of bodies are influenced by this condition. It appears now that we may add lubricating ability to the already long list of effects proceeding from this one prime cause."

The relation of colloid chemistry to industrial problems has attracted much attention of late, and in regard to lubrication Dunstan and Thole remark : "Recent work on colloid chemistry points to the fact that heavy oils must be included in the category of iso-colloids, *i.e.*, polyphase systems in which the disperse component is of the same chemical nature as the dispersion medium. Just as water must be regarded as a system in which molecules such as $(H_2O)_3$ co-exist with simple H_2O molecules, so in a lubricating oil the disperse phase is a molecular aggregate suspended in a dispersion medium of simpler and similar structure." They point out that the high viscosity, the large temperature-coefficient of viscosity and especially the hysteresis effects which have been observed in heavy oils are readily explained on this hypothesis. W. Ostwald* states that whenever a colloid comes in contact with a surface it fails to maintain a uniform special distribution. "On the surfaces of contact the concentration becomes different from that obtaining in the inner parts of the colloid mass. . . . Usually the colloid tends to concentrate on the surfaces." This is known as *adsorption*, and the phenomenon was first pointed out by Willard Gibbs. Hardy, in a recent Paper,* has called special attention to this. He finds that with a true lubricant a film of the order of $1\mu\mu$ thickness is sufficient to cause two glass surfaces to slide over each other with the minimum friction, and concludes "that lubrication depends wholly upon the chemical constitution of a fluid," and that "the true lubricant is always a fluid which is adsorbed by the solid face. If this be so, then the problem of lubrication is merely a special problem of colloid physics."

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* "Introduction to Theoretical and Applied Colloid Chemistry" (1917), p. 87.

That oiliness is not due to viscosity many experimental results show. In connection with the work of the Lubrication Committee of the Department of Scientific and Industrial Research, Messrs. Stanton and Hyde have made a large number of experiments on the friction of mineral and other lubricating oils with the Lanchester worm gear testing machine at the National Physical Laboratory. The results show conclusively that different mineral lubricating oils at the same viscosities give very different frictional results. The tests were made through a wide range of temperatures, and it was then possible to plot the results on a diagram having viscosities for abscissæ and ordinates friction. Fig. 1 shows one series of results obtained. It must be remembered that viscosity also affects the results here; but the property of oiliness shows itself strongly in the results.

It appeared to the writer that the property of oiliness might show itself most clearly when the two metallic surfaces were in actual contact, and that, therefore, valuable results might be given if the static coefficient of various oils between different metals were measured.

The friction experiments now to be described were made under conditions which insured that the surfaces should be in actual contact, and any differences in the frictional coefficients obtained with the oils tested must have resulted from the contact of the oily metals and not from the viscous flow of the liquid.

The experiments were made with a small hand-driven machine. Three pegs, each 5/32 in. diameter, rested upon the

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* "A Problem in Lubrication." "Journal" of the Society of Chemical Industry, Vol. xxxviii. (1919), p. 7T.

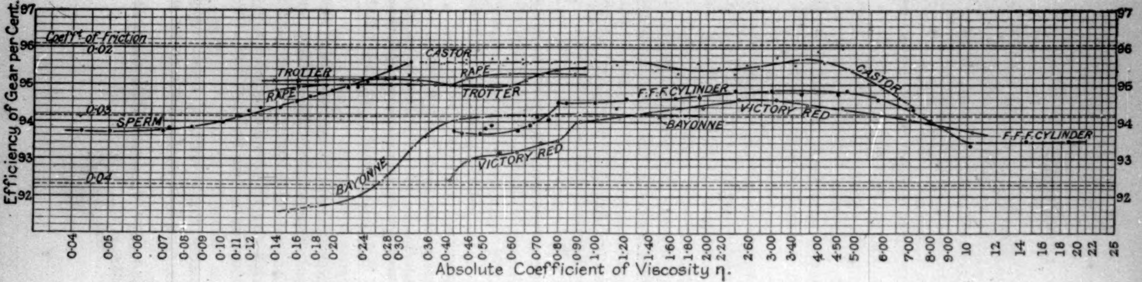
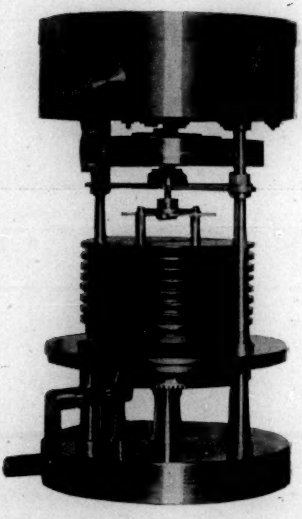


FIG. 1.—VARIATION OF EFFICIENCY OF WORM GEAR WITH VISCOSITY OF OIL.

Mean Distributed Load on Teeth of Gear = 1½ tons/sq. inch.
Mean Speed of Worm Shaft ... ... = 1080 revs./min.
Average Horse-Power Transmitted ... = 33½.



FRICTION TESTING MACHINE (DEELEY).

[*To face* p. 78.

flat surface of a disc of metal which could be slowly rotated. These pegs were secured, concentrically, to an upper disc which could be weighted as desired, and which actuated a spindle to which a spiral spring and recording finger were attached. When the lower disc was rotated, the pegs were carried with it by the friction until the surfaces slipped, owing to the stress on the spring, and the finger gave the value of the frictional resistance. To damp the oscillations of the finger, the spindle to which the finger and spring were attached was geared to a small train of wheels, the freely revolving end wheel of the series having a weighted rim to increase its inertia. A pawl and ratchet were also attached to the gear, to prevent the finger from suddenly falling back when the surfaces commenced to slip. By very slowly moving the driving handle the finger was soon brought to the position giving the static coefficient. The movable disc upon which the pegs rested lay in a circular dish, which could be filled with oil.

To insure clean surfaces, or surfaces as clean as possible, the rubbing metals were ground in water with flour of carborundum. They were then polished with fine wet emery cloth, rubbed well in water with a cork to remove as much emery as possible, dried with clean blotting paper, and finally well warmed to get rid of all traces of moisture. The temperature of the surfaces during the experiments varied from 16·0°C. to 18·0°C.

When the surface is thus prepared, and no lubricant is used, the static coefficient gets greater and greater as the surfaces continue to rub against each other. The initial static coefficient of a clean surface of mild steel resting upon cast iron was, in one instance, 0·154, and rose in the course of about 10 minutes to 0·417, and was still rising when the experiment was stopped.

A series of experiments was quickly made with surfaces of mild steel on cast iron, all quite clean. The results obtained are shown in Table I. The surfaces during each experiment were worked upon each other as little as possible, so as not to alter the conditions of their surfaces by wear.

Such clean surfaces are very sensitive to any contaminating agency. In one instance such a surface gave a momentary static coefficient of about 0·160. This surface was then

TABLE I.—*Clean Mild Steel upon Clean Cast Iron.*

Load per square inch. lbs.	Static coefficient of friction.	Kinetic coefficient of friction.
52·0	0·167	0·175
43·3	0·173	0·188
34·6	0·184	0·197
26·0	0·204	0·204
17·3	0·213	0·223
8·6	0·230	0·229

breathed upon for about 2 seconds, and a further test then gave a coefficient of about 0·452.

Each friction value given in the tables to follow is the mean of five determinations for each load of 52·0, 43·3, 34·6, 26·0 and 17·3 lb. per square inch respectively.

In the case of lubricated surfaces the procedure was as follows: As soon as the surface was clean and dry it was wetted with the lubricant to be tested, well rubbed with a cork to remove carborundum or emery, and then wiped clean with dry blotting paper. The surfaces were then worked in contact for a time, and the dark streak left on the disc was cleaned off with blotting paper and oil. The working of the surfaces together in this manner tended to smooth them, and it was found that the more this was done, with the surfaces dry but oily, the more nearly the static and kinetic friction approached each other in value. As the frictional value thus obtained varied with the extent to which the oil was rubbed off, no real value attaches to such figures; but the test was considered useful as it brought the rubbing surfaces into good condition. In Table II. some results obtained in this way are given.

TABLE II.—*Rape Oil with Mild Steel on Cast Iron. Surfaces Dry.*

Lbs. per square inch. lbs.	Coefficient of friction.
52·0	0·205
43·3	0·200
34·6	0·203
26·0	0·208
17·3	0·218
8·6	0·229

The static and kinetic coefficients were practically the same.

Table III. shows the static coefficient of friction and the efficiencies obtained for a number of oils when tested between mild steel and cast iron, and between mild steel and a lead bronze alloy of the following composition :—

Copper	85·88
Tin	5·71
Lead	4·00
Zinc	4·19
Iron	0·10
Nickel	0·12
	100·00

The numbers representing efficiency were calculated by the following formula :—

$$100-(\text{static coefficient}\times 100)=\text{Efficiency}.$$

TABLE III.—*Surface Flooded with Oil.*

Description of oil.*		Mild steel on cast iron.		Mild steel on lead bronze.	
		Static coefficient.	Efficiency.	Static coefficient.	Efficiency.
HB. clock oil	M.	0·271	72·9	0·275	72·5
Bayonne oil	M.	0·213	78·7	0·234	76·6
Typewriter oil	M.	0·211	78·9	0·294	70·6
Victory Rod oil	M.	0·196	80·5	0·246	75·4
F.F.F. cylinder oil	M.	0·193	80·7	0·236	76·4
Manchester spindle oil.	M.	0·183	81·7	0·262	73·8
Castor oil	V.	0·183	84·7	0·159	83·1
Valvoline cylinder oil	B.	0·143	85·7	...	...
Sperm oil	A.	0·127	87·3	0·180	81·1
Trotter oil	A.	0·123	87·7	0·152	84·8
Olive oil	V.	0·119	88·1	0·196	80·4
Rape oil	V.	0·119	88·1	0·136	85·4

* A=animal oil; V=vegetable oil; M=mineral oil; B=blended oil.

There is a marked difference between the friction of the various oils when mild steel is opposed to cast iron and the copper alloy respectively. Rape and olive oil give the best results, and mineral oil the worst, castor coming between the mineral lubricants and the rest. Rape and olive oil are of equal friction-reducing value between mild steel and cast iron, but rape is the better oil between mild steel and the lead bronze.

It is interesting to note that the static coefficient of those oils which stand lowest on the list and have the smallest coefficients of friction are those which have proved in practice the best lubricants.

The value of the coefficient of static friction as a test for lubricants depends upon whether care has been taken to properly clean the surfaces in contact. It was found that unless the last traces of water were driven off by heating the results were unreliable.

The coefficients of friction given in Table III. make it clear that the static friction varies, not only with the lubricant used, but also with the metals in contact. Oiliness would appear to be an effect produced by the lubricant upon the metallic surfaces with which it is in contact, rather than a property dependent upon any particular physical property of the lubricant.

It would appear that the unsaturated molecules of the lubricant enter into a firm physico-chemical union with the metallic surfaces, thus forming a friction surface, which is a compound of oil and metal. This solid surface would also appear, in the case of metallic surfaces, to be much more than one molecule thick, the oil either penetrating some little distance into the metal, and altering its physical properties, or, as a result of abrasion, forming a paste of metal plus oil between surfaces covered by oil layers one molecule thick. Even when dry, such a surface differs remarkably from a clean metallic surface. When the contaminated skin of a metallic surface has been removed by a file, the teeth of the file cut the metal and remove it easily; but if such a clean surface of metal be slightly oiled, or even if the hand be rubbed over it, the file will not cut anything like so freely. Thin films of matter of this kind cannot be wiped off. It is necessary to grind the surfaces under water, or to cut off a thin layer. When a metallic surface is clean, water easily wets it, and the moisture may be readily removed by heating.

Clean metallic surfaces in frictional contact soon tear each other and heat; but when a lubricant is interposed it unites with the metals forming a solid, which is itself a lubricant. If this substance, oil plus metal, should be torn off the surface, fresh oil combines with the freshly exposed metal. In this way the metallic surfaces are prevented from touching and seriously abrading each other. However, if the load be heavy, the heating resulting from the friction may be so great that the lubricant is destroyed, and the surfaces then seize and abrade each other.

Some experiments by Stanton and Hyde on the friction of mixtures of mineral oil with rape oil showed that a small proportion of rape oil had almost as marked an effect upon the friction as a much larger percentage. The rape oil, or a constituent of it, evidently displaced the mineral oil on the surfaces, and formed a rape oil plus metal film, which was as effective almost as if the surfaces had been in contact with pure rape oil.

Lubrication. *By* H. M. MARTIN, A.C.G.I.

NOTHING of fundamental importance has been added to the theory of lubrication as it came from the hands of Osborne Reynolds in 1886. Sommerfeld made certain much needed improvements in the mathematical treatment of the cylindrical bearing, and even more valuable work, both from the theoretical and practical standpoints, was accomplished by Mr. A. G. M. Michell of Melbourne in 1905, who showed that it was possible to extend Osborne Reynolds' theory to bearings of finite dimensions, whereas both Osborne Reynolds and Sommerfeld had had to assume that the bearings they discussed were of infinite length; or alternatively that leakage of lubricant from the sides of the brass could be neglected. Mr. Michell showed that this assumption was far from being satisfied in practice.

From Osborne Reynolds' theory it appears that the law of comparison between bearings is that, under similar geometrical conditions, the load carried per inch *run* of a bearing (measured transverse to the direction of motion) is constant, being independent of the absolute dimensions of the bearing.

In short, if W be the load carried per inch run, its value is given by the relation

$$W=\frac{\lambda U}{c^2}\varphi$$

Here λ denotes the viscosity of the lubricant, U the relative velocity of the opposing surfaces, and c a geometrical coefficient, whilst φ is another co-efficient, the value of which varies with the amount of side leakage.

For a flat block of unlimited length (measured transverse to the direction of motion) the value of φ is .159, the viscosity pressure and velocity being taken in C.G.S. units. For a square block Michell has shown that the value of φ falls to .067. The writer has calculated that in this case about one-third of the lubricant escapes at the sides of the block and two-thirds at the trailing end.

It is thus evident that the assumption that side leakage can be neglected, as made by Osborne Reynolds and Sommerfeld, is materially in error.

Incidentally, it may be mentioned, in passing, that in the Michell Patent Extension case tried last March, evidence was given to the effect that the adoption of the Michell Thrust Block by the Navy had resulted in a saving to the nation of £600,000 during the years of the war.

Although, as appears from the foregoing, some of Osborne Reynolds' numerical co-efficients may require correction to bring them in to accord with actual practice, the present writer believes that on the physical side he left the theory of lubrication complete. Amongst his noteworthy applications of it was the explanation he was able to give of the slipperiness of ice.

Reynolds held that in all cases of lubrication, even imperfect lubrication, the opposing solid surfaces were separated by a film of the lubricant, which kept them from coming into physical contact.

The present writer thoroughly endorses this view; at any rate, so far as fluid lubricants are concerned.

As for solid lubricants, it would seem that no one has yet been able to suggest a "working model" to represent their action. It is, however, conceivable that this may be due, at least in part, to such lubricants reducing to a common level the inequalities of the opposing surfaces, and that the actual lubrication is then effected by some fluid.

Leaving this question aside, as in need of further investigation, the problem naturally arises as to how it is that certain fluids are recognised as good lubricants and others as bad ones. Under favourable conditions, indeed, any fluid will act as a lubricant. Mr. Michell has used molasses to lubricate sugar-making machinery. Mr. Ferranti had, so the writer is informed, some amount of success in the attempt to lubricate spinning machinery with air, and water proves quite a satisfactory lubricant for the stern tubes of destroyers, the load carried being some 20 to 25 lb. per sq. in. of projected area, and the rubbing speed as much as 12ft. or 15ft. per second.

Nevertheless, as is well known, under conditions of imperfect lubrication some fluids prove satisfactory, and others entirely useless though credited with an equal or greater viscosity.

In a note contributed two or three years ago to the Committee on Lubricants and Lubrication, the present writer pointed out that in the case of imperfect lubrication the load was probably concentrated on a few " active " spots of limited area. A similar opinion had been expressed previously by Osborne Reynolds, who observed that under such conditions the friction would tend to be proportional to the load, which is not the case with well lubricated bearings.

If the load be thus concentrated the pressure in the oil film at the active spots will obviously be extremely high, and the writer accordingly suggested that under such conditions the viscosity of the lubricant might have a value very different from that obtaining at atmospheric pressure. He called attention to the work of Prof. Faust at Göttingen, who had found that the viscosity of such liquids as ether, carbon bisulphide and ethyl alcohol increased enormously as the pressure was raised from one to 3,000 atmospheres. He suggested accordingly that the Committee should find out whether anything similar occurred with lubricating oils. This suggestion was adopted, and the experiments made at the National Physical Laboratory, using for the purpose a very ingenious and simple apparatus devised by Dr. Stanton, and described by him at the recent meeting of the British Association.

These experiments did, in fact, show that at high pressures, the viscosity of oils was greatly increased. Had it turned out that this increase was higher with vegetable and animal than it was with mineral oils, some explanation of the property of " oiliness " might have been afforded. The reverse was, however, found to be the case, mineral ~~oils~~ showing a greater rise of viscosity with pressure than did animal and vegetable oils, and yet they are notoriously worse lubricants.

An explanation of this peculiarity must therefore be sought elsewhere.

Some 12 or 18 months ago Mr. R. M. Deeley informed the Lubrication Committee that he had found that good lubricants adhered much more firmly to metal surfaces than poor lubricants did. The present writer then suggested that under high rates of shear there might, with a bad lubricant, be slip at the boundaries of the oil film and the solid surfaces which it served to separate.

If this were the case lubrication would break down, as the moving journal would be unable to drag in the requisite supply of oil to maintain the film intact.

Some support for this view seems to be afforded by a remarkable experiment described by the late Lord Rayleigh in the "Phil. Mag." of January 1918. In this experiment a little carriage fitted with hemi-spherical feet of glass rested on a glass plate. The force required to produce sliding was measured under different conditions. With no lubricant, the surfaces being chemically clean, the resistance was high, whilst with a mere trace of lubricant probably approximating to a mono-molecular film, the resistance was remarkably low.

When, however, more lubricant was added so as to give a thick film, the resistance to sliding rose most markedly.

Now there is evidence that the attraction between a surface and the molecules of a mono-molecular layer of a liquid, is notably higher than the attraction between such a surface and a molecule of the liquid deeply buried under its fellows. For example, if a drop of oil be allowed to spread freely on water it will form ultimately a layer one molecule thick. If then the extent of the surface be decreased the excess of molecules, over and above those required to form a mono-molecular layer, gathers up into little lenses, the oil strives in fact to secure the largest possible extent of mono-molecular film. On this view, therefore, in Lord Rayleigh's experiment, the very thin layer of lubricant could withstand without slip a higher rate of shear, than could be sustained when the layer next the glass consisted of deeply buried molecules.

In practical cases of lubrication the rate of shear is often extremely high. The writer has calculated* that in turbine reduction gears, such as are used for ship propulsion, the thickness of the oil film separating the teeth is about $\frac{1}{8000}$ in., whilst the pressure in this film attains the value of some

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* *See* "Engineering," August 11, 1916, p. 120.

3 tons per sq. in. This calculation was based on a load of 520 lb. per linear inch, uniformly distributed over the whole length of the gear. As more than one tooth is in engagement the load per linear inch of the tooth amounted to about 267 lb. In actual fact with rigidly mounted pinions an equal distribution of the load cannot be maintained. The torque alters the pitch of the helices, so that even if a perfect bearing from end to end of the teeth were secured at say, ¼ load, there would no longer be a uniform bearing at full load. As a matter of fact it is found that the surface of the teeth actually flows in service. The pressure, therefore, in the oil film must at times attain to 30 tons per sq. in., and the thickness must, when this flow takes place, be correspondingly very much less than the $\frac{1}{8000}$ in. calculated on the hypothesis of an equable distribution.

The shear stress in the film must be exceedingly high. Under the hypothesis of uniform distribution of the load it comes out at 23 lb. per sq. in., and must be several times as great when the metal flows. Even when flow takes place lubrication is perfectly maintained. Indeed, Lord Rayleigh's experiment would seem to indicate, that lubrication is possible with films only a few molecules thick. In such cases the shear stress must be enormous, and there seems no inherent improbability that with bad lubricants slip does occur at the boundary between solid and fluid.

In the article "Atom," of the ninth edition of the Enc. Brit., Maxwell reports experiments in which slip was observed between the walls of a tube and a gas flowing through it. These experiments do not seem to have been confirmed, and no case has been reported of slip between a tube and a liquid. In none of the experiments, however, have the shearing stresses approached those which must occur in an imperfectly lubricated bearing.

In view of the fact now thoroughly established that metal will flow before seizure occurs it seems probable that in badly lubricated bearings, thoroughly run in, the load is concentrated on a narrow band of the brass, and not on a number of isolated spots as the writer formerly suggested. The initial irregularities in brass and journal will in fact tend to be levelled by the intense local pressures developed.

Molecular Layers in Lubrication. *By* H. S. Allen, *M.A., D.Sc., Physical Laboratory, The University, Edinburgh.*

In the " Notes on Lubrication," by Principal Skinner, and in the ensuing discussion (" Proc." Phys. Soc., Vol. 31, p. 94, 1919) emphasis was laid on the fact that under certain conditions vegetable and animal oils may be better lubricants than mineral oils. The best lubricants among the latter are those with a large proportion of unsaturated hydrocarbons. The property of oiliness is said to come in when there are regions of metallic contact. In this connection it may be of service to draw attention to the important work of Irving Langmuir on surface films (" Journ." Am. Chem. Soc., Vol. 38, p. 2221, 1916; Vol. 39, p. 1848, 1917). Adsorption of liquids or gases at the surface of a solid body is considered to be the result of unsaturated chemical forces at the surface. In many cases the adsorbed layer is only one molecule thick, and the conditions are determined by the shapes of the molecules and the relative activities of different portions of the molecules. The fact that animal and vegetable oils spread upon water whilst mineral oils do not, is explained by Langmuir as due to the presence of an " active group " in the molecule in the former case, and its absence in the latter. The soluble glycerine ester end of the rod-like molecule tends to dissolve in the water, but the insoluble hydrocarbon end refuses to do so, with the result that the molecule stands on end and the surface is covered with a layer of closely packed molecules of the animal or vegetable oil. On the other hand the mineral oils, with hydrocarbon groups at each end, refuse to spread. Similarly in the case of the adsorption of a liquid by a solid surface, when the molecules contain active groups, they will become oriented and will pack into the surface layer in much the same manner as in the case of oil films spread upon the surface of water. The application of Langmuir's theory to the case of lubrication is simple. In the extreme case we may picture each of the metallic surfaces covered with a layer of oil one molecule thick, resembling a piece of velvet firmly glued to the metal with the pile outwards. Two such velvet-clad surfaces can glide over one another with but little friction. Supposing that in some way actual metallic contact has been established, that oil will be most efficient which tends to spread over the metallic surfaces to form a lubricating layer. If the view here suggested be correct the property of "oiliness" depends neither on the viscosity nor on the compressibility of the liquid in the first instance, but on the chemical forces called into play between the active part of the oil molecule and the solid surfaces of the bearing. Thus "oiliness" would depend not only on the nature of the lubricant but also on the nature of the solid surface with which the liquid is brought into contact.

Remarks submitted before the Meeting by PRINCIPAL S. SKINNER, *M.A.*

THE two main divisions of lubrication are :—

(1) Full lubrication with an abundant supply of the lubricant such as in an oil bath.

(2) With a limited supply as is obtained with an oil pad with a little oil on it.

In division (1) the physical property of the lubricant which determines the absorption of energy is the viscosity of the liquid. A cylindrical vessel, which contains a liquid in which hangs a coaxial cylinder supported on a torsion wire, forms a well-known laboratory apparatus for measuring viscosity. The outer cylinder is rotated, and a torque is exerted on the suspended cylinder through the viscosity of the liquid. We have only to suppose that the axis of the inner cylinder is moved parallel to itself towards the outer cylinder in order to arrive at an arrangement similar to a wheel working on a fixed axle. The displacement of the inner cylinder causes forces which tend to push it back. The resultant forces are what supports the weight in a machine. The forces were fully considered by Osborne Reynolds in his theory of lubrication. The forces may modify the properties of the fluid as it is known that increased pressure increases the viscosity, whilst the tensional forces may produce a stretching of the lubricant and ultimately a rupture in it. It is this last aspect of full lubrication that suggested the necessity of knowing exactly what went on inside a bearing. For this purpose glass bearings have been examined with results that I have described elsewhere.

Recently one of my students has made experiments with glycerine in a Lahmeyer oil tester. Here we have a fully lubricated bearing. The torque is measured. As a result it was found that the torque obtained was of the order of that calculated from the viscosity. Special care was taken in these experiments to obtain the temperature of the lubricating film by means of a thermo-electric couple. This result bears out the view that viscosity is the important physical property in a fully lubricated machine.

Regarding division (2) lubrication with a limited supply of lubricant, we may trace the action of two solids rubbing on one another from the state of absolute cleanness to a state when there is a film of oil separating them. Some most interesting experiments have been made recently by Mr. W. B. Hardy and his son, and described and discussed in the "Philosophical Magazine" of July this year. These experiments, and their theory, may go far towards explaining the action of very thin films of oil. Briefly it appears that the clean glass surface of a watch-glass always seizes on a clean glass plate, however little the normal pressure between them. The watch-glass could not be displaced without tearing itself and the plate. When very thin films of liquids were used it was found that some liquids were inactive, whilst others promoted slipping and were lubricants. This division of liquids into active and inactive is an important discovery. The inactive liquids included water, alcohol, ether, whilst the active liquids included oils. With thick films all might be lubricants. As a result it appeared that oils in very thin layers could prevent seizing. It is only necessary, perhaps we may venture to say, to have the thinnest layer of active liquid to enable smooth metal surfaces to pass over one another freely. For lubrication with limited supply of lubricant we should, therefore, require a fluid that must readily form a film on the surface.

The theory briefly is that certain thin films will prevent the molecules constituting the surfaces from coming within their distance of cohesion, or from exerting cohesive attraction.

PROCEEDINGS AT THE MEETING.

PROF. C. H. LEES, F.R.S., President, in the Chair.

The PRESIDENT said they had met that evening to discuss the question of "Lubrication." The Council had thought that the time was opportune for such a discussion, and had accordingly called together engineers, physicists and chemists likely to be interested in it. They hoped to hear from the engineers what a lubricant ought to do and to be, and which were good lubricants and which were not; the physicists might be able to connect lubricating power with other physical properties of liquids, with viscosities, surface tensions and adsorptions; while the chemists might explain what was the connection between lubricating power and chemical composition of a lubricant.

Dr. T. E. STANTON, F.R.S., said that as the problem of lubrication tended to become more and more one for the organic chemist, he was afraid he was not a suitable person to open a discussion on that subject, but it might be of interest to the Society if he called attention to a few of the outstanding difficulties.

In the first place, they must realise that the work during the last 40 years had, to a large extent, been concerned with one case of lubrication—two lubricating surfaces in relative motion under moderate loads and with a limited supply of lubricant. Their practical knowledge of the laws of lubrication under these conditions was due mostly to Mr. Beauchamp Power. He had a half bearing and a 4 in. journal, and the lower part of the journal was always immersed in oil. His experiments ranged as regards loads from 100 lb. to 525 lb., and from 100 to 450 revs. per min. He found the frictional resistance fairly independent of the load and nearly proportional to the velocity, and that it always depended very considerably upon the temperature of the oil film. All these results were consistent with the assumption that the journal and shaft were separated by a film of oil, and on actually testing it Power discovered the existence of a film of oil under considerable pressure, and on measuring its intensity at different points he found that the total pressure was equal to the load.

After these results were published, the matter was taken up by Osborne Reynolds, who realised that this film might be thick enough for the unknown boundary conditions between the two surfaces to disappear, in which case the equations of hydro-dynamics could be applied to the motion. As the result of a somewhat laborious investigation he generally obtained an approximate solution for the half bearing. This theory showed that there must be essentially a difference of radii between the journal and the bearing, and that the point of nearest approach was on the off-side of the journal. The speaker illustrated this by reference to diagrams.

Reynolds also found when the distance between the surfaces was half the difference in radii, that then the pressure all through the film was everywhere greater than atmospheric pressure; and when the load exceeded the value corresponding to this condition, the pressure on one side became less than atmospheric pressure and the film tended to rupture.

This mathematical theory of lubrication was taken up by Sommerfeld, who had written a long Paper on the subject, pointing out several errors in calculation in Reynolds' Paper; also by Michell and by Harrison. Harrison's Paper was interesting because he had shown that when the bearing was treated as a complete cylinder an exact solution in two dimensions was obtained very easily.

An interesting fact brought out by Harrison's analysis was that the moment on the journal was not the same as the moment on the shaft, and on actually working the values out he found that taking that case of Reynolds, where the shortest distance between the bearing and shaft was half the difference of the radii, the ratio of the moment on the shaft to the moment on the journal was 1·5, which was disconcerting to those who, like the speaker, had to do tests of lubricants. To sum up for this particular case of lubrication his own opinion was that the theory had been so well worked out and the agreement between the observed facts and the calculated results was so good that more research was not needed.

It was the other case where the surfaces were so close together that the unknown boundary conditions were predominant that gave them trouble. As an instance of that he would take the case of a worm-gear transmitting high powers. In the case of the journal on the shaft, the oil film broke down at 120 lb. per square inch. Mr. Hyde in his experiments had found that in these worm gears, when they were transmitting 50 H.P., and the pressure between the teeth was of the order of 5 tons per square inch, they could get an efficiency over 97 per cent. He might call attention to those experiments Mr. Hyde had carried out, though he thought, perhaps, all the members of the Society were familiar with them. Mr. Hyde had tested a series of oils, animal, vegetable and mineral, in the worm-gear testing machines designed by Mr. Lanchester, and the results were interesting, but not easily explained. The speaker showed the results of a few of these in the form of a diagram, in which the values of the efficiencies obtained were plotted on a temperature base.

It was of interest to point out that the viscosity of the oils could have had no appreciable effect, for the reason that between temperatures of from 25°C. to 45°C. they could get no difference in efficiency worth mentioning, whereas in some cases the ratio of the viscosities was as great as 6 to 1. Perhaps one of the most interesting features of the tests was the effect of mixing a vegetable oil, such as rape oil, with mineral oils. It did not seem to matter whether they used 20, 15, 10 or 2½ per cent., they still obtained the same kind of effect, which consisted of a definite horizontal shift of the critical point through a range of temperature of about 20°C.

As regards the nature of the dependence of the efficiencies on the actual values of the viscosities, Mr. Hyde devoted a series of experiments to that subject and found that the change of viscosity with the pressure was very large indeed, but still failed to throw any light on the peculiar behaviour of the oils in the worm-gear testing machine.

Mr. A. S. **Barnes**, who read Mr. R. M. Deeley's Paper (*see* pp. 1s to 11s), said that he was deputising that evening for the author, who was unfortunately not in sufficiently good health to be able to come and speak. Mr. Deeley began his Paper by reflecting that friction was both the friend and the enemy of engineers. There were many aspects of friction on which the engineer depended for use and assistance, but it was the hostile aspects of friction with which they were dealing that night, and those were countered or made as slight as possible by the use of lubricants.

Mr. J. H. **Hyde** gave a demonstration of Mr. Deeley's instrument. He said that Mr. Deeley's machine was designed to measure the static and kinetic coefficients of friction under various conditions of oiling. The metals could either be scantily lubricated or flooded with oil. Of the two metals to be tested together, one was in the form of a disc and the other in the form of three small pegs attached to a frame. In the lower part of the machine there was a shallow dish rotated by means of a small handle. Into this dish the metal disc and frame were placed, the latter such a way as to be free to rotate. On turning the small handle the three pegs were carried round against the torsion of a spring. The spindle of that spring was connected to a pointer through a train of wheels which provided a damping on the pointer. The wheel travelling at the highest velocity had a weighted rim to affect this.

Mr. Deeley had explained in his Paper that it was absolutely necessary to get rid of the contaminating film on the surfaces, so that it was found to be essential before consistent results could be obtained that both the disc and the three pegs should be ground under water. The greatest care must be observed that no oil, or even a contaminating film from the hands, should get on either surface before they were placed in position on the machine. On first putting the surfaces together and rotating the handle, the co-efficient of friction was usually fairly high, but on continuing to rotate them in oil the friction became higher and higher the longer the experiment was carried on. On pouring some oil in the dish and then beginning to rotate, the pointer was carried round until the slipping began to take place. The oil film was then formed between the surfaces and the pointer rushed back a considerable distance.

Mr. Hyde demonstrated the various parts of the mechanism.

Mr. **Leonard Archbutt**, F.I.C., F.C.S., said: In the consideration of lubrication problems it is necessary to make a very clear distinction between two prevailing conditions, viz. :—

(1) Those in which the solid surfaces are completely separated by a film of oil, such as occurs in the lubrication of cylindrical journals and their bearings, properly fitted, revolving at high speed and supplied with abundance of oil; and

(2) Those in which, owing to the shape or condition of the surfaces, the kind of motion, the high bearing pressure, low speed, or an inade-

quate supply of oil, the oil-film cannot form completely, or becomes broken, and the solid surfaces come into contact.

In (1) the friction is entirely due to the viscosity of the oil, as shown by Osborne Reynolds. Engineers have Reynolds' theory to guide them in the design of machines, and the chemist, by determining the physical and chemical characteristics of an oil, is able to give sound advice as to the suitability of any particular oil to meet the required conditions.

In (2), the circumstances are quite different. Oils alike in viscosity are found to have quite different friction-reducing values, and we have no knowledge to guide us in the selection of oils other than that gained by practical trial under working conditions. Experience has shown that mineral oils, as a rule, reduce friction less than animal and vegetable oils, why we do not know, though the great difference in the capillarity of the two kinds of oil is of importance. Of two mineral oils, or two fixed oils, alike in viscosity, one cannot say, without trial, which will give the lower friction. What we can say is, that under these conditions, *i.e.*, where solid friction comes in, viscosity alone is not the only or the most important factor, and that some other unknown property which we term "oiliness" is very important. An outstanding problem in lubrication is to determine the nature and cause of this property of "oiliness." Mr. Deeley's ingeniously designed machine seems likely to give us a valuable means of measuring oiliness, and his theory that oiliness is the property possessed by an oil, or a constituent of it, of combining with the surface to be lubricated and forming a lubricating film is, I think, in accordance with the theory of Irving Langmuir, of which Dr. Allen has given such a clear account. Dr. Allen says: "The application of Langmuir's theory to the case of lubrication is simple. In the extreme case we may picture each of the metallic surfaces covered with a layer of oil one molecule thick, resembling a piece of velvet firmly glued to the metal with the pile outwards." In a memorandum to the Lubricants Committee some time ago Mr. Deeley wrote: "I consider that the oil molecules which adhere to the metallic surfaces form part of the solid and that we have solid friction between such oily surfaces when they touch. . . . One comes to regard (lubricated) metallic friction surfaces as being of the nature of velvet pile rather than of unyielding material."

The importance of the friction surface is shown in the case of the white alloys known as "Anti-friction metals," and has long been familiar to engineers. In some recent tests of a mineral lubricating oil under conditions in which there was solid as well as viscous friction, I found that under the same conditions of speed and pressure, and with the same oil, bearings lined with white metal would carry double the load carried by bronze bearings without any increase of the friction. Prof. Goodman many years ago noticed that the presence of traces of bismuth and other metals had a great effect in modifying the friction of a white metal surface. He found that if the added metal had a smaller atomic volume than the white alloy the friction was increased, but if it had a greater atomic volume the friction was diminished.

On Dr. Langmuir's theory the molecule of the oil and of the surface are held together by residual or secondary valence, and the more chemically active the lubricant the more firmly it will be attached. Mr.

Southcombe's discovery of the importance of the free fatty acid in lubricating oils is, therefore, of great interest. Hitherto, free acid of any kind in a lubricant for metallic surfaces has been regarded as a constituent to be avoided, and for very good reasons. In steam cylinders, in lubricating greases containing water, and even in lubricating oils containing no more moisture than that due to natural causes, fatty acids attack the metal surfaces forming metallic soaps which obstruct steam passages and dissolve in and thicken the lubricant, besides damaging the metallic surfaces themselves, but Mr. Southcombe has shown that fatty acids in very restricted amount are an advantage, as they improve the lubricating value of a mineral oil to the same extent as a much larger quantity of fatty oil would do. Here, again, it is necessary to be quite clear as to the conditions under which the fatty acid is advantageous. In condition (1), where the friction is entirely due to the viscosity of the oil, the fatty acid is of no advantage. It is in condition (2), where solid friction occurs, that its presence is useful. The effect is very marked and very remarkable. In some experiments made with a Thurston machine under conditions of low speed and fairly high bearing pressure—7 ft. per minute and 270 lb. per square inch—I have found that 1 per cent. of rape oil fatty acid added to a mineral lubricating oil reduced the coefficient of friction from 0·0047 to 0·0035. Two per cent. of fatty acid did not reduce the coefficient to a lower figure. Neutral rape oil added to the mineral oil also reduced the friction, but 60 per cent. of the neutral oil was required to give as low a friction coefficient as 1 per cent. of the free fatty acid. Perfectly neutral rape oil is not, of course, a commercial article, perhaps fortunately. All commercial rape oil contains free fatty acid and, consequently, much less is required than if it were neutral, but the important fact is that a very small percentage of fatty acid added to a mineral lubricating oil improves the lubricating value of the mineral oil as much as a much larger percentage of fatty oil and leads to considerable saving in the cost of oil, as well as other advantages. It is important to note, however, the fact that I have found perfectly neutral rape oil to give a lower friction than a mineral oil of the same viscosity and that the presence of free fatty acid in rape oil does not seem to reduce the friction nearly so much as rape oil fatty acid added to mineral lubricating oil reduces the friction of the mineral oil.

I have made comparative tests with commercial rape oil containing 2·4 per cent. of free fatty acid, perfectly neutral rape oil prepared from the same rape oil, and a mineral oil of very nearly the same viscosity at the temperature of the tests. I found that the neutral rape oil gave a much lower friction than the mineral oil, the acid rape oil gave a somewhat lower friction than the neutral rape oil. The figures obtained were:—

Mineral oil (900/7)	0·0078
Rape oil, neutral	0·0050
Rape oil, 2·4 per cent. free acid	0·0045

If these results are confirmed, say, by careful tests on Mr. Deeley's machine, it appears that the oiliness or lubricating efficiency of the unsaturated molecules of the neutral rape oil is nearly as great as that of the free fatty acid molecules, but that in their influence on the hydro-carbon molecules of the mineral oil the acid molecules are much more active. The size of the molecules and the thickness of the molecular layer will have an important influence.

Langmuir's Papers will, no doubt, be carefully studied by physical chemists. They appear likely to throw very considerable light upon this subject.

In a communication, received by the Secretary on December 4, 1919, Mr. **Archbutt** wrote that since the meeting he had read the Paper by Mr. W. B. Hardy and Flight-Lieut. J. K. Hardy in the "Phil. Mag." of July, 1919, in which their theory of static friction and lubrication is very fully and clearly discussed. He thought this important Paper should be read in connection with the present discussion. It was interesting to note from the table of "threshold values," on page 44, of Messrs. Hardy's Paper that between glass and glass oleic acid gave the same low friction as castor oil. He agreed that it was essential to experiment with individual chemical substances in investigating the theory of lubricants, but it was also necessary to experiment with commercial lubricants acting between such dissimilar metals as are used in the construction of machines, in order to obtain the information which engineers required to guide them in the lubrication of machinery. He looked forward with great interest to further Papers on this subject from Mr. Hardy and his son.

Principal S. **Skinner** said he would to some extent depart from the synopsis of his printed remarks, just noting that he had divided lubrication into two parts, the full lubrication on the one hand, and the very thin film on the other. With regard to the first division, he was going back to an old subject, *i.e.*, the cavitation in a fully lubricated bearing. He was having constructed a bearing and journal with a good fit, and was hoping to show the rupture that he had observed with a similar piece of apparatus in which the fit was not so good. He felt that it was certain that viscosity was the chief property which a fully lubricated bearing had to fight against, still it was of interest to see as far as possible what was exactly going on in a bearing. He had a slide in the lantern which showed the cavitation. (The speaker here showed his slide). He would like to mention that the tensile strength of fluids had been the subject of some careful experiments by Max Toepler, who had studied the rupture of liquids in an apparatus consisting of a lens oscillating on a glass plate.

The second division of lubrication which he gave, referred chiefly to Mr. Hardy's experiments with very thin films, to which should be added the views and experiments of Langmuir. In connection with films against solids he had noticed recently an interesting letter in "Nature" from Mr. Twyman, who asked whether anyone had noted the change of the refractive index in liquids against a surface. He might congratulate himself on the synopsis of his printed remarks, since it agreed largely with what had been said already by other speakers.

Mr. W. B. **Hardy**, F.R.S., said that he had attempted to reduce the problem to its simplest elements by avoiding purely mechanical complication and by the use of undivided chemical substances as lubricants.

The influence of chemical constitution upon lubrication cannot be traced by the study of mixtures. It is impossible to assign to any particular substance an absolute value as lubricant. For instance, water is an anti-lubricant for glass, a feeble lubricant for bismuth and an admirable lubricant for ebonite. Acetic acid and tripropylamene are admirable lubricants for glass and only feeble lubricants for bismuth. Lubrication is due to the interaction of two substances at a surface. The property known as "oiliness" is due to the influence of the lubricant upon the surface energy of the solid. It is to be expected, therefore, that lubrication can be defined only in terms of the chemical constitution of the lubricant *and* of the solid to which it is applied.

The study of a large number of chemical individuals proves that the influence of chemical constitution is complex beyond expectation. The only generalisation to which no exception has yet been found is that the introduction of a *single* hydroxyl group into a molecule increases its action as a lubricant. A second and a third hydroxyl group, however, diminish lubricating value. In general the single hydroxyl group is very much more effective attached to a ring or closed chain nucleus than to an open chain nucleus.

The worst lubricants are simple symmetrical compounds, such as benzine and carbon tetrachloride. Putting H in place of Cl, or CH_3 in place of H increases lubrication. There is no particular virtue in unsaturation. In simple compounds, such as paraffins, it slightly increases lubrication. In other cases it may decrease lubrication. In most chemical series lubrication increases as the molecular weight increases, according to a law which may be exponential or linear. In some series it decreases as molecular weight increases. An example is the simple aliphatic esters.

The most striking result is that lubrication has no particular relation to the fluidity of the lubricant. Thus in the fatty acid series or paraffins the curve connecting lubricating power with molecular weight shows no deviation where the substances change from fluids to solids at the temperature of observation. It must be remembered that these experiments deal with static friction, but it is not possible to avoid incidental observations upon the simplest case of kinetic friction. They support the view that the fluid state has no particular value in kinetic friction as such and apart from purely mechanical factors.

The above conclusions are from experiments with bismuth sliding on bismuth. They will not hold for glass on glass. For instance, the hydroxyl group does not always increase lubrication, and it is almost without influence compared to the carboxyl group, when the solid is glass.

Professor F. G. Donnan said that he was at present engaged upon investigation of the optical analysis, by means of measurements of the double refraction produced, of the strain and rate of relaxation of strain in lubricating oils and other liquids and mixtures of liquids subjected to high rotational shear. He quite agreed as to the importance of the surface-energy factor, but it appeared to him that in a complete theory of lubrication this other aspect of the problem might be of equally great importance.

Mr. J. E. Southcombe said he was sure the discussion they had had that evening had been most inspiring, and they were agreed definitely over two points; first, that there were two distinct types of lubricating phenomena, one in which the mechanical phenomena were dealt with, and those in which they got low speeds and high pressures, and in which the truly physical aspects were the more important.

Being somewhat of an enthusiast on the subject of "oiliness," it was particularly interesting to him to find the meeting was so unanimous in considering that "oiliness" was something very distinct, and a very important phenomenon in the practice of lubrication. It was this phenomenon which Mr. Wells and he commenced to investigate some years ago. They were primarily interested in trying to distinguish wherein fatty oil differed as a lubricant from mineral oil. The considerable superiority of the fatty oils over the mineral oils—the pure hydro-carbon oils—was very important in some types of machinery.

The first thing that would obviously appear as an essential qualification of a lubricant was that it should wet the surface. He would like to draw attention to a bearing and a journal in a condition of rotation which was slightly eccentric. Suppose they put in between two surfaces some mercury. In that case they would get a meniscus. The effect of that meniscus was to draw the liquid out in the directions of less constriction. The definition of a lubricant was that it was a substance which should flow into the regions of higher pressure. If, instead of mercury, oil were put in that space between the two surfaces, a meniscus of a different form would be obtained. In the first case the ends of the meniscus were directed outward, and in the second case they were directed inward, and by capillary laws the tendency was for the liquid to move into the more restricted areas of high pressure. The vital condition of lubrication, therefore, was surface tension.

There was a further point. Let them consider the rise of liquid in a capillary tube. If the distance of constriction was less than the normal capillary rise, it followed that the oil would tend to lift the journal, and would extend itself. It would be true to regard water as a lubricant if there were always a pure, clean surface, but not in the case of a contaminated or greasy surface. The water meniscus in a greasy bearing would be the other way round.

Having recognised that that peculiar capillary phenomenon seemed to be the most important thing in lubrication, attention must be directed to getting some sort of measure of the capillarity of oils, to see whether they could distinguish some property in the fatty oil not possessed by the mineral oil, since density and viscosity and compressibility did not give a measure of lubricating value.

To make a long story short, they seemed to find themselves forced to the conclusion that there was some phenomenon which remained to be investigated with regard to surface tension. Instead of employing the old methods in investigating surface tension, they were endeavouring to determine the surface tension of oils against another liquid, in one case water. The speaker described on the blackboard his experiments with a pipette filled with oil, and showed the results obtained with mineral oils, and then with certain fatty oils, including olive oil, cocoanut oil, lard, and a series of others. In the case of the mineral oils, the results in the pipette were of the order of 98 to 102 drop numbers, and in the case of the fatty oils were of the order of round about 150, or of a totally different order to the drop numbers obtained in the case of the mineral oil. They had, therefore, at once some distinguishing characteristic between the fatty oil and the mineral oil. Ultimately those values of that variation of the drop number were shown to rise from the presence of fatty acids. Recently, Prof. Lewis had repeated that experimental work and determined the capillarity of oils against mercury surface.

It remained to ask what exactly the *modus operandi* was in the practical case of lubrication. He did not find himself at all in agreement with Mr. Hardy when he said that the problem of lubrication should be considered on pure surfaces and surfaces free from all disturbing influences; because it was the disturbing influence or the contaminating effect that was of importance above all others in practical lubrication. When it came to be considered, oil was only rarely used under ideal conditions. In nine out of ten cases, water was splashing about and metallic particles, dirt, dust, oxides and so on were present. The greatest problem with those of them who were oil manufacturers was to make an oil that would emulsify with water for a certain job and for another job would not emulsify with water—in other words, to deal with the contaminating effect.

If they considered a brass surface in practical use, a little reflection would show that it was not a brass surface; it was a surface of metallic brass coated with some contaminating agency. If there was a film of moisture on that brass surface and they placed upon it a drop of oil, *the oil, supposing it contained a fatty acid, would have a lower interfacial tension and would tend to spread.* A film of metallic oxide on that bearing surface would behave similarly. In one sense that brought them to Langmuir's view. They must agree that where there was a tendency for chemical action, the surface tension necessarily became lower. If they had an inert hydro-carbon oil on such a surface, it had no chemical affinity in any sense whatever for the journal.

He did not propose to discuss that evening the number of details as to the practical application to known cases. The hypothesis had been worked out on a friction machine and in practice with most conclusive results. There was, of course, a further reason which supported the conclusion, viz., the adsorption layer. If they had between journal and bearing a pure hydro-carbon, there could be no possibility of adsorption. There must be a third substance capable of adsorption by the bearing surface.

Mr. E. Edser said that during the last few years he had been engaged in an industry of a most romantic character. It was concerned with the recovery of minerals from rocks. A rock might contain about 1 per cent. of copper in the form of sulphide, and the problem was to recover that copper. Under ordinary conditions the cost of getting it out of the rock would be greater than the worth of the substance. It was necessary to concentrate the copper to more than 10 or 15 per cent. That was done by grinding up the rock fairly finely so that it passed a screen of 60 to 80 mesh, then mixing it with water containing small quantities of oil or other substances, and then agitating the mixture violently. The result was that the mineral rose to the surface in a froth. Some of the small bubbles obtained were so persistent that they could be pressed between two microscope slides and on release would assume their original forms.

There were a multitude of problems in connection with that subject, none of them particularly difficult in itself. It was essential to learn under what conditions a particle would adhere to a bubble. A few years ago there was a good deal of literature published on that subject, especially in the United States, and all manner of effects were described and theories proposed. He wanted to point out that although flotation did not seem to have any connection with lubrication, it would probably be discovered on further investigation that a connection existed. There was this peculiar fact to reckon with, that the substances which oil adhered to, water would not adhere to. A substance seemed to have "affinity" either for water or for oil, but not for both. It seemed to him that the problem of lubrication involved the adhesion of oil to a shaft or to a bearing. An attempt to run a steel shaft in a steel bearing produced unsatisfactory results; different substances must almost invariably be used for the shaft on the one hand, and for the bearing on the other.

He would like to mention a point which was seldom given sufficient prominence, viz., that a surface layer of any substance whatever, solid or liquid was less dense than the interior of that substance.

The speaker illustrated on the blackboard his conception of the attraction and repulsion of molecules, and said that it was easy enough to imagine how oils and such like substances could become entangled in the surface layers of solids. In this connection he would like to mention an experiment performed by Clerk Maxwell, about 1870, in a lecture at the Royal Institution. He took a sheet of glass and a lens and measured the force exerted between them, finding that in order to get optical contact one must be pressed against the other with considerable force, and that on removing the force the lens moved away from the glass to a distance comparable with the wave-length of light. It was a significant fact that when two flat surfaces of glass were placed together the black spot indicating optical contact was never obtained, although the surfaces when left pressed against each other ultimately adhered and could not be separated.

Perhaps this phenomenon was due to the surface layer of a solid being in a comparatively rarified condition, somewhat resembling a gas; the surface layer of a liquid had been proved by Van der Waals to have the character described and there should be no essential difference between the surface layer of a solid and that of a liquid. It would be remembered that Beilby had proved that the surface layer of a solid was fluid and by gently rubbing a copper surface with chamois leather, films had been formed over small pits in the surface.

When two perfectly plane surfaces were placed "in contact" there appeared to be almost perfect freedom of motion of one over the other, which again suggested the rarefaction of the surface layers. A drop of water between the surfaces destroyed the freedom of motion, and a drop of oil did the same. This indicated the difficulties encountered in explaining lubrication; it is practically impossible to obtain effective lubrication between two perfectly flat surfaces. This fact was emphasised by Osborne Reynolds in his explanation of lubrication.

The **President** said he thought on the whole they had learned more about how the different liquids stood with regard to "oiliness" than about the reason one liquid was a good and another a bad lubricant. When he came there he hoped that some theory of adsorption such as that of Langmuir was going to offer a good explanation of all the facts, but that hope had been disappointed owing to the new facts brought forward by Mr. Hardy.

They had to thank the various speakers who had taken part in the discussion and the Secretary, Dr. Owen, who had organised the discussion so well.

COMMUNICATED REMARKS.

Mr. A. E. **Berriman**, Chief Engineer, Daimler Company: It has been made quite clear that there are two principal states of lubrication, which, for convenience of reference, may be called A and B; state A being the condition in which an intermediate stream of oil entirely separates the bearing surfaces; state B being the condition in which the bearing surfaces are apparently in solid contact.

From the engineering standpoint, state A is ordinarily associated with bearing pressures that do not exceed about 400 lb. per square inch, although higher values may be maintained in this state by special means under suitable conditions. State A has no coefficient of friction, since the friction under practical loads is sensibly independent of the load, but varies as the bearing area and the square root of the speed.

State B has a coefficient of friction, which experiments with the Lanchester machines show to be in the order of 0·025. It is distinguished from state A by this fact and also by the fact that wear does take place even in ideal conditions. The validity of regarding state B as a lubricated state depends on the fact that this wear does not destroy the surface nor does it generate more heat than the system can dissipate. In fine, it is a stable state.

The present discussion is an attempt to elucidate the physical condition of state B, and there are two principal theories under consideration. The Deeley theory contends that the metal surface is chemically contaminated by the action of the oil, so that a new solid compound is formed on the surface. The Martin theory contends that the metal is not chemically affected, but that good lubricants adhere to the surface in ultimate films.

It is to be observed that in both the Martin and the Deeley theories the phenomena are related to layers of monomolecular thickness, but in the one case the partial failure of lubrication is attributed to the tearing away of the contaminated metallic surface and in the other to a slipping of the ultimate oil film, as a prelude to metallic contact.

So far as the practical results are concerned, it matters perhaps little to the engineer what theory is proved to be correct; but it is of very first-class importance that the problem should be solved in such a manner as to provide engineering science with the data needed to ensure the successful operation of those moving parts that must, by their very nature, operate in state B.

The most that I, as an engineer, can contribute to the discussion is to recapitulate some of the practical phenomena bearing on the subject.

It is, for instance, common knowledge in engineering that certain materials are more difficult to lubricate effectively than others because they tend to score and seize more readily, but this does not in itself give the key to the reason.

It has also been well established that the bearing pressure in state B lubrication may attain a value at which the metal will flow, and yet changing the oil will cause a change in the coefficient of friction.

Yet another phenomenon is that a perfectly made plug gauge can be moved readily in a perfectly made ring gauge, if the surfaces are lubricated and the motion is maintained, but seizure will take place if the relative motion ceases.

Of these examples, the first two might be explained by either theory, but perhaps more readily by Deeley's than Martin's. The last-mentioned example of the plug and ring gauge, however, seems to be capable of explanation more readily by Martin's theory than Deeley's, because in this case there has been no physical abrasion to remove the chemical compound from the surface, and the quality of its friction should not be dependent upon a state of motion as appears, in this instance, to be the case.

The significant difference that takes place when the state of motion changes to a state of rest in this experiment suggests that oiliness is related to surface tension.

F. W. LANCHESTER, M.Inst.C.E.: During the time I have been actively engaged in engineering work, at least as far back as 1889, I have been acquainted with the fact that the lubricating properties of an oil depend upon factors other than viscosity; I believe that the millwrights and engineers of last century were as a class fully alive to the same fact. It is to be recalled that when mineral oils were first introduced and offered as lubricants such oils were not well received by the practical men of the day, and for a long while they were a drug on the market at a price far below that of other oils of similar "body."

My own experience carries me to a date a few years later than the publication of Prof. Osborne Reynolds' Paper on "The Theory of Lubrication." One of the results of that Paper was undoubtedly to throw discredit on the opinions of the practical man of that period, and to cause the alleged superiority of animal and vegetable oils to be regarded as moonshine. A secondary consequence was that about that period also mineral oils began to obtain wide acceptance as substitutes for animal and vegetable oils more or less implicitly on the results of viscosity tests. A contributory factor to the ready acceptance of mineral oils was, of course, the advent and general use of the internal combustion engine, for which most of the lubricants previously used were unsuitable.

My own experience, however, as to the inferiority of mineral oils, about 1890, was quite decisive. I found myself in charge of the test bench at a works manufacturing gas engines, and it was my invariable practice, if a bearing ran hot when first on test, to substitute castor oil for the mineral oil usually employed, when all difficulty at once vanished. The large end connecting rod bearing in particular was frequently doped in this manner. I tried other oils, including mineral oils of equivalent viscosity, but castor oil invariably gave the best results. This and other experience left no doubt whatever in my mind that there was a property of *oiliness* inherent to a greater or lesser extent in good lubricants that could not be accounted for by the ordinary physical properties or tests. I am quite sure also that this remained the view of the older hands and practical men of that date.

My worm-gear testing dynamometer was designed and built at the Daimler Works and was first put to work in 1912. This dynamometer gives readings that are estimated to be within one-tenth of 1 per cent. of the truth, and must be regarded, of its class, as an instrument of very high accuracy. The immediate result demonstrated was, of course, the high efficiency obtainable by a worm gear of good design, but the trials also brought out in a startling manner peculiarities in the properties of the lubricants employed. Not only did two oils of the same, or nearly the same, viscosity often give entirely different readings, such, for example, as a mineral oil in which the coefficient of friction might be nearly double that of an animal or vegetable oil, but the curves given by different oils were so characteristic that with a little experience it was almost possible to say what oil was used from an inspection of the plottings of the efficiency readings. It was quite evident at once that, compared to any previous oil-testing appliance, such as Thurston's machine, the new dynamometer possessed properties of an altogether higher order of merit.

Without giving or attempting a quantitative definition of *oiliness*, it is certain that qualitative differences may be recognised with certainty between different oils by aid of the worm-gear dynamometer. In truth, after working with this machine and having had through my hands the results of a great many tests and experiments, I was led to state that the lubricating properties of an oil for worm gear could no more be judged by its viscosity than the brew of beer or vintage of champagne. Thus the following two conclusions are given ("Proceedings" of the Institution of Automobile Engineers, Session 1912-13, Vol. VII.):—

1. That in general mineral oils are inferior to animal or vegetable oils.
2. That the viscosity of the oil is little or no guide in the selection of an oil for the purpose in question*; one might as well try and judge the merits of beer or any other beverage by the viscosimeter.

I am sure that to-day it is impossible to dispute the existence of a property of *oiliness* apart from the ordinary physical properties of a lubricant and, so far as this view is only of recent acceptance amongst scientific men, so far we may take the period from 1886 until to-day as a measure of the time it has taken to recover from an overdose of Osborne Reynolds. I have, of course, nothing to say against the work of Osborne Reynolds; his Paper or something equivalent was necessary to the scientific theory of lubrication, but I well remember the time when, mainly, I think, in consequence of his Paper, the old practical worker of last century, whether engineer or millwright, was looked upon as an ignorant and prejudiced person, and when those who shared his view were liable to be stigmatised as charlatans and quacks.

The fact is, that although Osborne Reynolds' Paper on "The Theory of Lubrication" is the truth and nothing but the truth, at least we may take it as such, it is certainly not the whole truth. It would, I think, be well to qualify it by referring to it as "The Theory of the Thick Oil Film in Lubrication." The whole question of the thin oil film and the conditions of lubrication pertaining to same, one might say an entire world of lubrication, is ignored. The conditions of lubrication as between gear teeth, whether it be spur-gear or worm-gear, in no way come within the scope of Mr. Osborne Reynolds' investigation, and moreover, the ordinary journal bearing is more dependent upon the oiliness of the lubricant than might be inferred. In the event of a journal bearing not being well bedded in the first instance the thin film lubrication and the oiliness of the lubricant are of paramount importance until, if we may so express it, the Osborne Reynolds' condition is established. The same applies in the event of a journal bearing becoming out of truth either from an accidental cause or ill-treatment.

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* Worm gear lubrication.

It must not be forgotten, however, that a great deal of knowledge of the behaviour of lubricants existed before Mr. Beauchamp Towers' experiments or Mr Osborne Reynolds' Paper thereon. Some physicists have, perhaps, a tendency to assume ignorance on the part of those who do not read Papers before Scientific Societies. Knowledge and the expression of knowledge are two entirely different things, and the one very commonly exists long before the other. I think there is no doubt that the engineers of 1850 and thereabouts understood perfectly well that it was necessary to maintain a *film* of oil or lubricant and that the thickness of the oil (which was later termed viscosity) was one of the factors which determined the suitability of an oil for its purpose. It is not easy to find explicit references, but Rankine, writing a decade or more before Osborne Reynolds' Paper, says:—

"Three results in the preceding Table, No. 16, 17 and 18, have reference to smooth firm surfaces of any kind, greased or lubricated to such an extent that the friction depends chiefly on the continued supply of unguent and not sensibly on the nature of the solid surfaces; and this ought almost always to be the case with machinery. Unguents should be thick for heavy pressures that they may resist being forced out and thin for light pressures that their viscidity may not add to the resistance."

Generally speaking, the writers of the older works on engineering took the question of lubrication for granted. It is rarely, if ever, for example, that in the index the word "oil" or "lubrication" finds any place, and one may more often than not search for a drawing or figure of such a thing as a lubricator in vain. However, references are found to the "oil film," showing that this essential conception of what we may describe as the unit of lubrication had been clearly formulated.

The various explanations given in the present discussion as to what in fact constitutes the property of oiliness have, of course, an interest quite apart from and beyond the mere establishment of the fact itself. If oiliness were a physical property one cannot but feel that it ought to be capable of definition and able to be expressed in terms of the dimensions L M.T.; it is, however, clearly by no means necessary that it should be a physical property. When under examination by a recent Committee (Institution of Mechanical Engineers) I expressed the view that the unknown property might possibly be due to the mechanical destruction of the oil by attrition, as, for example, as though the oil were decomposed, gasified or partially gasified by local heat; in such a case the actual lubricant might be microscopic bubbles of gas or a gaseous film, in other words—the oiliness might possibly not be due to the oil at all but to its products of dissociation. This was only put up as a suggestion and not as an opinion, my idea being that in the investigation of oiliness our minds must be kept open very wide, even to include a possibility so remote.

In many of the explanations put before the meeting the chemistry of the lubricant is being made the *pivot* of investigation and the suggestion of the saturation or non-saturation of the molecule is shown as a possible or probable factor of importance. The chemistry of the subject is not much in my line, but I do feel that the investigation should be encouraged and every hopeful channel should be thoroughly explored. If a reasonably complete theory of the lubricating property of lubricants can be developed we may be on the way to obtain lubricants of vastly greater potentiality than anything with which we are now acquainted; perhaps, for instance, lubricants that will enable machines to be worked at really high temperatures. Even should investigations only lead us to the more economic and appropriate use of those lubricants we at present know, the total money saving and financial value the world over would be incalculable.

R. S. WILLOWS, M.A., D.Sc.: Most theories of lubrication have concerned themselves chiefly with the physical properties of the lubricant, such as the viscosity. I think a successful theory must take into account certain molecular effects which, by some workers, are classed as chemical, but which I prefer to regard as physical, although I doubt whether there is any reason for differentiating between the two classes of forces. The present note is a contribution to that end.

One point to be explained is how the lubricant maintains a continuous film between the two rubbing surfaces. This is really the fundamental question why some liquids wet a given solid, while others fail to do so. A theory which has been advanced from different standpoints by Thomson ("Phil. Mag.," May, 1914) and Langmuir ("Jour. Amer. Chem. Soc.," 1917, *et. seq.*) goes some way towards a solution and has a great amount of evidence in its favour from other directions. In certain molecules some of the atoms are supposed to have lost and others to have gained one or more electrons (Thomson); hence these molecules will exhibit the properties of electrical doublets, and will be surrounded by a strong electrical field. Thomson shows, from measurements of dielectric constants, that certain chemical groups, like OH, COOH, &c., are especially marked in this respect. The cohesion of a liquid then arises, in part at all events, from the atttractions between such doublets; surface tension and viscosity are other manifestations of the same effect. To extend this to metals: It is well known that the electrons in metals are in a state of considerable freedom, hence when a liquid whose molecules possess fields of force comes near a metal it may have the power to bind certain of the electrons from the metal, so creating an electrical double layer, whose negative portion is near or in the oil. We have, in fact, what is usually called adsorption taking place. (If the molecular aggregates in the metal also possess a field the effect will be enhanced.) With such an arrangement considerable cohesive forces between solid and liquid wil be called into play; at the same time, and owing to similar causes, the viscosity of the oil will be large. Thus two of the main characters of a good lubricant—adhesion to the solid and high viscosity—can be accounted for, and such friction as there is will be between the adsorbed layer and those liquid layers near it. Langmuir's theory contemplates similar forces, although he does not state explicitly how they arise. Because they are localised in certain directions he classes them as chemical—in view of modern physics, not a very convincing argument—and ascribes them to secondary valencies—a name which does not make the matter clearer, at least, to a physicist. He has, however, shown by some strikingly beautiful experiments that the presence of such groups as OH, &c., does cause one liquid to spread over another, and has made some remarkable deductions concerning the shapes of certain molecules. Hardy has shown that a pure hydro-carbon oil does not spread on water, while an impure one spreads rapidly; while in a discussion on a Paper by Skinner last session it was stated that a saturated hydro-carbon is a poor lubricant, but its lubricating properties are greatly improved by the presence of unsaturated hydrocarbons. These facts are consistent with the theory given above, since, from the data given by Thomson and Langmuir, the saturated molecules will have weak fields of forces. It also explains the fact that animal and vegetable oils are better lubricants than those of mineral origin, since unsaturated molecules are more likely to be present. It would be interesting to know if the non-spreading property of a pure hydro-carbon oil is affected by heat owing to possible molecular changes causing doublets, and if the viscosities of a non-spreading and of a spreading oil differ appreciably. Perhaps Mr. Hardy can give us this information.

Experiments of the following type might also be useful. (1) Immerse a weighted quantity of powdered bearing metal in a good lubricating oil, then centrifuge strongly and weigh again, thus obtaining the weight (and thickness ?) of the adherent layer. Repeat with a non-lubricating but equally viscous liquid and so determine whether the adsorption is very different in the two cases. (2) The cohesion of a liquid can be determined by O. Reynolds' method. Theoretically it should be greater under a strong electric field, if Thomson's doublets are present, for they will align themselves under the action of the field, and neighbouring molecules will attract each other. Similar considerations would perhaps lead us to expect that good lubricating liquids should have high dielectric constants.

When molecules possess long chains, Langmuir shows that the non-active part of the molecule is turned from the interface water-compound when a thin layer of the compound spreads over the water surface; the other molecules of the compound will then have little attraction for the adsorbed layer, since they can approach only the inactive ends of the molecules in this layer. If such conditions occur in a lubricant we should have an adherent layer on the solid and reduced friction against the next liquid layer. This would ensure good lubrication, and the slipping between the liquid layers might be abnormally easy where the total thickness is only a few molecules.

VII. *First Steps in the Experimental Analysis of a Galvanic Cell.* By PROF. W. M. COLEMAN.

RECEIVED APRIL 8, 1919.

FIRST, to test the effects of changes of pressure in a galvanic cell upon the current it yields, I designed the following apparatus :—

A glass tube 3 cm. in diameter was used for the cell ; it was about 40 cm. in length, and one end was drawn down to a diameter of 4 mm. The cell was made inverted, and wires from the lower ends of the zinc and copper plates were led out at the bottom through a perforated rubber stopper.

A piece of rubber tubing was placed on the small upper end of the tubular cell, and the battery fluid was introduced by means of partial exhaustion with jet pumps.

The rubber tube was then connected with a bicycle pump, and the pressure was raised 240 mm. Hg., as shown by a manometer ; or it was lowered by about one-fourth of an atmosphere by connecting with a jet pump.

It was found that slowly raising or lowering the pressure seldom caused variation in current ; but a sudden rise of about 240 mm. Hg. caused the current to increase about 5 milliamperes, and a like decrease brought an equal decrease in current strength (shown by an average decrease in reading of 5 milliamperes)—from 65 to 60 milliamperes.

A rise of pressure of 240 mm. Hg. caused a slight fall in potential, a fall being noticeable in the scale of the D'Arsonval galvometer at each stroke of the pump, the total movement of the indicator amounting to three marks on the scale (equivalent, perhaps, to one-hundredth of a volt).

My next test was to find whether any effect would result from varying the pressure inversely between the zinc and copper plates. For this test I dissected the cell, and placed an element at each end of a glass tube 3 cm. in diameter and 2 metres long. Tests were made with this tubular cell held in a horizontal position ; also in the vertical position with copper and zinc ends alternately down. With a cell 2 metres long the change in hydrostatic pressure on reversing the ends was a little more than one-fifth of an atmosphere. The current increased about 2 milliamperes (the readings were 4 and 6 milliamperes) when the zinc end was suddenly turned down ; and it decreased when the tube was reversed.

In the above tests with the dissected cell, as well as those with the undissected but inverted cell, the absence of change in current when the pressure was changed slowly and gradually, and the changes in current with sudden changes of pressure indicate that the results are due to changes in polarisation by dislodgment of gas, and that they were not due to direct effect of changes of pressure upon the chemical action.

I expect to repeat the tests of potential already given, and to use a quadrant electrometer instead of a galvanometer.

I wish to acknowledge with thanks the assistance of Mr. W. H. Hayle, of the Cavendish Laboratory, in the foregoing experiments. I wish especially to thank Sir J. J. Thomson for the privilege of using the laboratory.

ABSTRACT.

When the pressure in a galvanic cell is suddenly varied, the current in the external circuit is altered, an increase in pressure increasing the current, and vice versa. When the pressure in the neighbourhood of the zinc electrode is suddenly increased, and that in the neighbourhood of the copper electrode is diminished at the same time, an increase in current takes place, and vice versa.

VIII. *Radiation from a Perfectly Diffusing Circular Disc.* (*Part I.*)

By JOHN W. T. WALSH, *M.A.* (*Oxon*), *M.Sc.* (*Lond.*). (*From the National Physical Laboratory*).

RECEIVED OCTOBER 30, 1919.

Synopsis.—When investigating the light emission from the positive craters of arcs, especially of the large ones of modern design, it becomes of importance to know what percentage of the total light is obscured by the negative carbon, particularly when the lamp is burning at a short arc length, which may be a fraction of the diameter of the positive carbon.

It is generally assumed that the positive crater radiates as if it were a perfectly diffusing circular disc, *i.e.*, a disc the radiation from which, in any direction, is proportional to the cosine of the angle between that direction and the normal to the surface.

To a first approximation it may next be assumed that the negative carbon acts as a second circular disc coaxial with the positive crater, and at a distance from it equal to the arc length.

On this assumption it is easy to find the proportion of the total light which is obstructed by the negative and this may be done either (i.) by deducing the required result from Sumpner's theory of the whitened sphere, or (ii.) by calculation from first principles.

The first method is shorter; but the second has been added here as it serves to introduce a method which leads to a closer approximation to the actual conditions of the problem, viz., the light obscured by an infinitely long negative carbon.

It will be seen that the results given have a wider application than that for which they were obtained, as the problem of determining the illumination at any point of a disc by a second coaxial perfectly diffusing disc is one of considerable interest in photometry. The amount of light received by a searchlight mirror from the positive crater of an arc also follows at once. The illumination at any point in the penumbra produced by a disc shading a coaxial radiating disc, can also be calculated. This is shown in a second part of the Paper.

Dr. Sumpner[*] has shown that if the interior of a sphere consist of a perfectly diffusing surface (*i.e.*, a surface radiating according to the cosine law), then the flux from any element of the surface of the sphere is equally distributed over the remainder of the surface.

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[*] Physical Society, "Proc." XII., 1892, p. 10.

Now suppose a diffusing plane surface be introduced into the sphere, as at *AB* in the figure. It is known that the radiation to any point from this surface will be the same as that from the segment of the sphere, *ACB*, which it encloses, for if the intrinsic brilliancy of the two surfaces be the same, their brightnesses viewed from any point will be equal.

Hence it follows that a plane circular surface *AB*, uniformly bright, produces a uniform illumination over the surface of any sphere of which its edge forms a small circle.

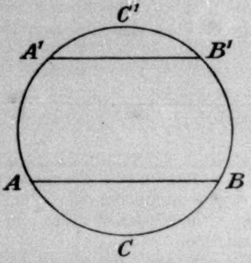

FIG. 1.—PRINCIPAL SECTION OF RADIATING SPHERE AND DISCS *AB*, *A′B′*.

Thus the proportion of the total flux from *AB* which is received by a parallel disc *A′B′* is equal to the ratio of the spherical area *A′B′C′* to the whole spherical area *AA′C′B′B*.

Now if *m* is the radius of the sphere, *a* the distance between *AB* and *A′B′*,

$$2r=A'B'$$

$$2R=AB.$$

Then $$a=\sqrt{m^2-r^2}+\sqrt{m^2-R^2}.$$

Also, spherical area $A'C'B'=2\pi m(m-\sqrt{m^2-r^2})$.

Spherical area $AA'C'B'B=2\pi m(m+\sqrt{m^2-R^2})$.

Total flux emitted from *AB* is $\pi R^2 . \pi\Phi$, where Φ is the flux emitted in the normal direction per unit area.

$\therefore$ **Flux received by** $A'B'$ is $\dfrac{m-\sqrt{m^2-r^2}}{m+\sqrt{m^2-R^2}}\pi^2R^2\Phi$

$$=\frac{\pi^2\Phi}{2}[(a^2+r^2+R^2)-\sqrt{(a^2+r^2+R^2)^2-4r^2R^2}].$$

It is clear from what has been said above that the same amount of flux will be received by any disc of the same radius as $A'B'$ placed so that its edge forms a small circle of the surface $AA'C'B'B$.

The results obtained above can also be deduced from first principles as follows :—

Let A be an element of a perfectly diffusing surface (*i.e.*, a surface which radiates according to the cosine law), so that if the flux radiated normally be Φ per unit area, the flux radiated in a direction making an angle θ with the normal is $\Phi \cos \theta$ per unit area.

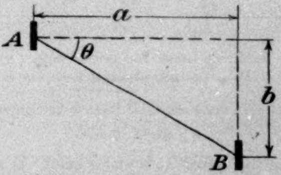

FIG. 2.—FLUX RECEIVED FROM RADIATING ELEMENT A BY A PARALLEL ELEMENT B.

If now B be an element of a surface parallel to A and at a normal distance a from it, then if the line AB make an angle θ with the normal at A, the intensity of the radiation falling on B from A is given by

$$\frac{A(\Phi \cos \theta)}{a^2\sec^2\theta}.$$

Hence the flux received by B from A is

$$\frac{A\,\Phi \cos \theta}{a^2\sec^2\theta}\,.\,B \cos \theta=\frac{A.B.\Phi}{a^2}\cos^4\theta.$$

(I.) From this result it is easy to find *the flux received from a radiating disc by an elementary area parallel and coaxial with it.* For the flux received by B from the radiating disc is (*see* Fig. 3), if R be the radius of the disc,

$$\frac{B\Phi}{a^2}\int_0^R \cos^4\theta \,.\, 2\pi b \,.\, db \quad (\text{where } b=a\tan\theta)$$

$$=2\pi B\Phi\int_0^{\tan^{-1}R/a} \cos\theta\sin\theta d\theta.$$

$$=\pi B\Phi R^2/(R^2+a^2)$$[*].

Note.—Similarly it may be shown that the flux received by the whole of a disc of radius r from an elementary radiating

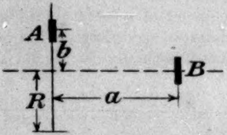

FIG. 3.—FLUX RECEIVED FROM RADIATING DISC OF RADIUS R BY A PARALLEL AND COAXIAL ELEMENT B.

area A parallel and coaxial with it at a distance a is equal to

$$\pi A\Phi r^2/(r^2+a^2).$$

Since the total radiation from the element is $\pi A\Phi$ (obtained by putting r infinite compared with a), the fraction of the whole which falls on the disc is $r^2/(r^2+a^2)$.

(II.) The above result can now be used to find *the flux received from a radiating disc by an elementary area parallel with it but at a distance ρ from the axis.* For let PQS (Fig. 4) be the radiating disc of radius R, and let its centre be C, and the projection upon its plane of the elementary area, O. Let $OC=\rho$ and let OPQ be drawn cutting the circle in P and Q so that the angle $COQ=a$.

The values of OP and OQ are given by the expression

$$\rho\cos a\mp\sqrt{R^2-\rho^2\sin^2 a}.$$

Now the flux reaching the elementary area from the area $PQQ'P'$ of the disc (where $QOQ'=da$) is the difference between that reaching it from a sector OQQ' and that from a sector

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[*] This expression was obtained by E. P. Hyde, "Bull." Bureau of Standards, 1907, III., p. 88.

OPP′, *i.e.*, it is, from the result of section (I.), if *B* be the area of the element,

$$\frac{d\alpha}{2\pi}(\pi B\Phi)\left\{\frac{OQ^2}{OQ^2+a^2}-\frac{OP^2}{OP^2+a^2}\right\}.$$

Hence the total flux received from the whole disc is

$$\frac{B\Phi}{2}\int_{\sin^{-1}-R/\rho}^{\sin^{-1}R/\rho}\frac{4a^2\rho\cos\alpha\sqrt{R^2-\rho^2\sin^2\alpha}}{(a^2+R^2+\rho^2\cos 2\alpha)^2-4\rho^2\cos^2\alpha(R^2-\rho^2\sin^2\alpha)}d\alpha.$$

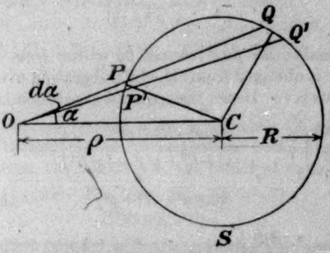

Fig. 4.—Flux Received from Radiating Disc *PQS* by a Parallel but Non-Coaxial Element whose Projection on the Plane of *PQS* is *O*.

Putting $(R^2-\rho^2\sin^2\alpha)=x^2$ and writing c for $(a^2-R^2+\rho^2)/2a$ this becomes

$$B\Phi\int_0^R\frac{x^2dx}{(c^2+x^2)\sqrt{R^2-x^2}}$$

$$=B\Phi\left[\sin^{-1}\frac{x}{R}\right]_0^R-\frac{B\Phi c}{\sqrt{c^2+R^2}}\left[\cos^{-1}\frac{c}{R}\sqrt{\frac{R^2-x^2}{c^2+x^2}}\right]_0^R$$

$$=\frac{\pi B\Phi}{2}\left[1-\frac{c}{\sqrt{c^2+R^2}}\right].$$

Hence the radiation received is

$$\frac{\pi B\Phi}{2}\left\{1-\frac{c}{\sqrt{c^2+R^2}}\right\}^{*}\quad\text{where } c\equiv(a^2-R^2+\rho^2)/2a.$$

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[*] This expression was obtained by P. D. Foote, "Bull." Bureau of Standards, XII., 1916, p. 585.

NOTE.—Similarly it may be shown that the flux received by a disc of radius r from an elementary area parallel with it but at a distance ρ from its axis is given by

$$\frac{\pi A\Phi}{2}\left\{1-\frac{c'}{\sqrt{c'^2+r^2}}\right\},\ \text{where}\ c'\equiv(a^2-r^2+\rho^2)/2a.$$

Since the total radiation from the area is $\pi A\Phi$ (obtained by putting r infinite compared with a and ρ), the fraction of the whole flux from the area which falls on the disc is given by

$$\frac{1}{2}\left\{1-\frac{c'}{\sqrt{c'^2+r^2}}\right\}.$$

(III.) Finally, *the flux received by a disc from a radiating coaxial disc* is obtained from (II.) by integrating over the circle whose radius is r. Hence the required expression is

$$\frac{\pi\Phi}{2}\int_0^r\left\{1-\frac{a^2-R^2+\rho^2}{\sqrt{(a^2+R^2+\rho^2)^2-4\rho^2R^2}}\right\}2\pi\rho d\rho.$$

Putting $x\equiv(a^2-R^2+\rho^2)$ this becomes

$$\frac{\pi^2\Phi}{2}\int_{\rho=0}^{\rho=r}\left\{1-\frac{x}{\sqrt{x^2+4a^2R^2}}\right\}dx$$

$$=\tfrac{1}{2}\pi^2\Phi[(R^2+r^2+a^2)-\sqrt{(R^2+r^2+a^2)^2-4r^2R^2}].$$

Hence the flux received by a disc of radius r from a coaxial radiating disc of radius R at a distance, a, from it is

$$\tfrac{1}{2}\pi^2\Phi[(R^2+r^2+a^2)-\sqrt{(R^2+r^2+a^2)^2-4r^2R^2}].$$

Since the total radiation from the disc of radius R is $\pi^2\Phi R^2$ (obtained by putting r infinite compared with a and R), the fraction of the whole flux from the disc which falls on the disc of radius r is

$$[(R^2+r^2+a^2)-\sqrt{(R^2+r^2+a^2)^2-4r^2R^2}]/2R^2.$$

Since this expression is symmetrical with respect to r and R, clearly one disc receives from a second coaxial disc the same amount of flux that the second disc would receive from the first.

The following applications of the foregoing expressions are of interest.

(IV.) For a given radiating disc, to find the surface traced out by the edges of coaxial discs receiving equal amounts of flux from it.

Let the given amount of flux be Q.

Then $\quad Q=\frac{1}{2}\pi^2\Phi[(R^2+r^2+a^2)-\sqrt{(R^2+r^2+a^2)^2-4r^2R^2}]$

Hence the equation to a principal section of the required surface is

$$(R^2+x^2+y^2)-\sqrt{(R^2+x^2+y^2)^2-4x^2R^2}=2Q/\pi^2\Phi=K^2, \text{ say,}$$

if the axis of the discs and a radius of the radiating disc be taken as axes of y and x respectively. This reduces to

$$(R^2+x^2+y^2)^2-4x^2R^2=(R^2+x^2+y^2-K^2)^2,$$

i.e., $\quad x^2(4R^2-2K^2)-2K^2y^2+K^2(K^2-2R^2)=0.$

But $\quad K^2=2Q/\pi^2\Phi$ and $Q<\pi^2\Phi R^2$.

$$\therefore \quad K^2<2R^2.$$

Hence the principal section is an hyperbola—*i.e.*, the surface required is an hyperboloid of revolution, about the common axis of the discs, the equation to the principal section being

$$(2x^2-K^2)(2R^2-K^2)-2K^2y^2=0,$$

where $\quad K^2\equiv 2Q/\pi^2\Phi.$

Any two discs parallel with the radiating disc and bounded by this surface will receive the same amount of flux, so that any zone (or any portion of a zone, by symmetry) of this surface must receive equal amounts of flux on both sides, for the flux passing *in* must balance that passing *out* through any part of the surface.

(V.) To find the amount of flux received from a radiating disc by an element of any sphere of which the edge of the disc forms a small circle.

If m be the radius of the sphere,

$$a=\sqrt{m^2+r^2}+\sqrt{m^2-R^2}.$$

$$\therefore \quad a^2+r^2+R^2=2m^2+2\sqrt{(m^2-r^2)(m^2-R^2)}.$$

The flux received per unit area by a zone of the sphere, parallel with the disc and at a distance a from it, is (Fig. 5)

$$\frac{\pi^2\Phi}{4\pi m}\frac{d}{da}[(R^2+r^2+a^2)-\sqrt{(R^2+r^2+a^2)^2-4r^2R^2}]$$

$$=\frac{\pi^2\Phi}{2\pi m}\cdot\frac{d}{da}[R^2+a(\sqrt{m^2-R^2}-m)]$$

$$=\pi\Phi[\sqrt{m^2-R^2}-m]/2m$$

which is independent of a, and therefore constant over the sphere. [This is the result obtained from first principles by Dr. Sumpner (*loc. cit.*).]

(VI.) It is clear that if O be the projection of an element of the radiating disc on the plane of the base of the cylinder PTQ, the flux received from O by the walls of the cylinder will be equal to that received by the area bounded by the two tangents from O to the circle and the portion of the circumference convex to O.

This equals, as in section (II.),

$$\frac{A\Phi}{2}\int_{\sin^{-1}-r/\rho}^{\sin^{-1}r/\rho}\frac{OP^2}{OP^2+a^2}da$$

$$=\frac{A\Phi}{2}\int_0^{\sin^{-1}r/\rho}\left[\left(\frac{OQ^2}{OQ^2+a^2}+\frac{OP^2}{OP^2+a^2}\right)-\left(\frac{OQ^2}{OQ^2+a^2}-\frac{OP^2}{OP^2+a^2}\right)\right]da.$$

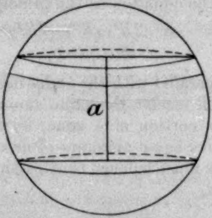


FIG. 5.—FLUX RECEIVED PER UNIT AREA BY A ZONE OF THE SPHERE OF WHICH THE RADIATING DISC FORMS A SMALL CIRCLE.

The second half of this integral has already been evaluated in section (II.) as

$$\frac{\pi A\Phi}{4}\left\{1-\frac{a^2+\rho^2-r^2}{\sqrt{(a^2+\rho^2-r^2)^2-4\rho^2r^2}}\right\}.$$

The first half

$$=\frac{A\Phi}{2}\int_0^{\sin^{-1}r/\rho}\left.\frac{a^2(OP^2+OQ^2)+2OP^2OQ^2}{(OQ^2+a^2)(OP^2+a^2)}\right\}da$$

$$=\frac{A\Phi}{2}\int_0^{\sin^{-1}r/\rho}\left[1+\frac{t^4-a^4}{(t^2-a^2)^2+4a^2\rho^2\cos^2 a}\right]da.$$

Putting $(t^2-a^2)\tan\alpha=x$, this becomes

$$\frac{A\Phi}{2}[\alpha]_0^{\sin^{-1}r/\rho}+\frac{e\Phi}{2}(t^2+a^2)\int_{\alpha=0}^{\alpha=\tan^{-1}r/t}\frac{dx}{x^2+(t^2-a^2)^2+4a^2\rho^2}$$

$$=\frac{A\Phi}{2}\left[\sin^{-1}\frac{r}{\rho}+\frac{(t^2+a^2)}{\sqrt{4a^2\rho^2+(t^2-a^2)^2}}\tan^{-1}\frac{r(t^2-a^2)}{t\sqrt{4a^2\rho^2+(t^2-a^2)^2}}\right].$$

Hence the flux received by the walls from the whole radiating disc is equal to

$$\frac{\Phi}{2}\int_r^R 2\pi\rho\sin^{-1}\frac{r}{\rho}d\rho+\frac{\Phi}{2}\int_r^R\frac{2\pi\rho(t^2+a^2)}{\sqrt{4a^2\rho^2+(t^2-a^2)^2}}\tan^{-1}\frac{r(t^2-a^2)}{\sqrt{4a^2\rho^2+(t^2-a^2)^2}}d\rho$$

$$-\frac{\Phi}{2}\int_r^R\pi^2\rho\left\{1-\frac{a^2+\rho^2-r^2}{\sqrt{(a^2+\rho^2-r^2)^2-4r^2\rho^2}}\right\}d\rho.$$

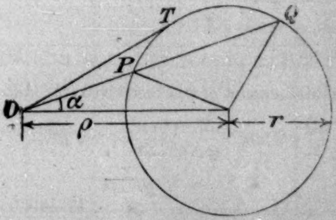


FIG. 6.—FLUX RECEIVED FROM A RADIATING DISC BY THE WALLS OF A CYLINDER INFINITELY LONG IN ONE DIRECTION.

The first integral

$$=\frac{\pi\Phi}{2}\left[R^2\operatorname{cosec}^{-1}\frac{R}{r}-\frac{\pi r^2}{2}+\sqrt{R^2r^2-r^4}\right].$$

The second integral (since $\rho^2=r^2+t^2$, so that $\rho d\rho=tdt$) becomes

$$\pi\Phi\int_{\rho=r}^{\rho=R}\frac{t^2+a^2}{\sqrt{(t^2+a^2)^2+4a^2r^2}}\tan^{-1}\frac{r(t^2-a^2)}{t\sqrt{(t^2+a^2)^2+4a^2r^2}}tdt.$$

Integrating by parts and putting $y\equiv\sqrt{(a^2+t^2)^2+4a^2r^2}$, we have that this

$$=\frac{\pi\Phi}{2}\left[y\tan^{-1}\frac{r(t^2-a^2)}{ty}\right]_0^{\sqrt{R^2-r^2}}-\frac{r\pi\Phi}{2}\int_{\rho=r}^{\rho=R}\frac{y^2-2t^2(t^2-a^2)}{(r^2+t^2)(a^2+t^2)}dt$$

$$=\frac{\pi\Phi}{2}\left[y\tan^{-1}\frac{r(t^2-a^2)}{ty}-(a^2+r^2)\tan^{-1}\frac{t}{r}-4ar\tan^{-1}\frac{t}{a}+rt\right]_0^{\sqrt{R^2-r^2}}$$

$$=\frac{\pi\Phi}{2}\left[\sqrt{(R^2+r^2+a^2)^2-4r^2R^2}\tan^{-1}\frac{r(R^2-r^2-a^2)}{\sqrt{(R^2-r^2)\{(R^2+r^2+a^2)^2-4r^2R^2\}}}\right.$$

$$+\frac{\pi}{2}a\sqrt{a^2+4r^2}-(a^2+r^2)\tan^{-1}\sqrt{\frac{R^2}{r^2}-1}$$

$$\left.-4ar\tan^{-1}\frac{\sqrt{R^2-r^2}}{a}+r\sqrt{R^2-r^2}\right].$$

The third integral

$$=\frac{\pi^2\Phi}{4}\int_r^R\left\{1-\frac{a^2+\rho^2-r^2}{\sqrt{(a^2+\rho^2+r^2)^2-4r^2\rho^2}}\right\}d\rho^2.$$

This equals, as in section III.,

$$\tfrac{1}{4}\pi^2\Phi[(R^2-r^2)+a\sqrt{a^2+4r^2}-\sqrt{(a^2+r^2+R^2)^2-4r^2R^2}]$$

Thus, the total amount of flux received *by the walls* is

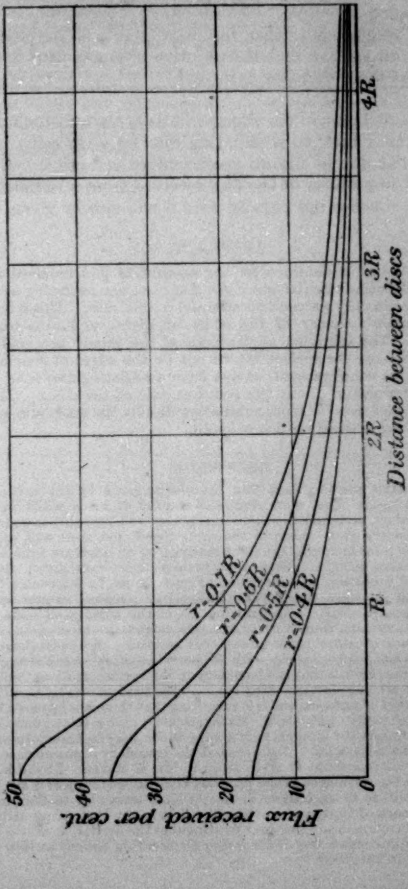
$$\frac{\pi\Phi}{2}\left[\frac{\pi Q}{2}+Q\tan^{-1}\frac{r(R^2-r^2-a^2)}{Q\sqrt{R^2-r^2}}+2r\sqrt{R^2-r^2}\right.$$

$$\left.-(a^2+r^2+R^2)\tan^{-1}\frac{\sqrt{R^2-r^2}}{r}-4ar\tan^{-1}\frac{\sqrt{R^2-r^2}}{a}\right],$$

where $Q\equiv\sqrt{(a^2+r^2+R^2)^2-4r^2R^2}$.

As an example of the practical application of the formulæ given above, it is of interest to take the case of a carbon arc of diameter D with negatives of diameters $(0{\cdot}4)D$, $(0{\cdot}5)D$, $(0{\cdot}6)D$, and $(0{\cdot}7)D$. The proportion of light obscured by a "disc" negative of these dimensions at various arc lengths is easily calculable from the formula of section (III.), and the results are shown in graphical form in Fig. 7.

The extra light obsecured by the stem of the negative in two particular cases has been calculated from the formula of section (VI.). For a negative whose diameter is half that of the positive, the obscuration of the stem of the negative is



Flux received per cent.
50
40
30
20
10
0
R
2R
3R
4R
Distance between discs
r=0·7R
r=0·6R
r=0·5R
r=0·4R

FIG. 7.

$1{\cdot}6_3$ per cent. for an arc length D and $4{\cdot}4_5$ per cent. for an arc length $D/2$.

These results show that for most practical purposes it is sufficient to assume that the negative obscures as if it were a disc of the same diameter as its end.

In a second part of the Paper will be given a formula for the flux reflected back to a radiating disc by a diffusing parallel and coaxial disc of known coefficient of reflection. A treatment of the problem of the flux received from a radiating disc by a disc which is not parallel with it will also be given.

ABSTRACT.

An expression is obtained for the amount of flux received from a perfectly diffusing circular luminous disc (*i.e.*, one radiating according to the cosine law) by a second parallel and coaxial disc. This is obtained from Sumpner's theory of the whitened globe, and also from first principles. The following applications of the theory are made: (I.) It is shown that the surface traced out by the edges of coaxial discs which receive equal amounts of flux from a radiating disc is an hyperboloid of revolution about the common axis of the discs. (II.) The amount of flux received from a radiating disc by the walls of a cylinder infinitely long in one direction is found.

DISCUSSION.

Mr. T. SMITH was surprised that the results given by the author were regarded as novel. They were almost all cases of problems which had been discussed before, certainly in other connections, and probably in connection with photometry also. Some of the cases considered were well known in optical guise; for instance, the light received by an aperture from a plane extended source of light. They were, perhaps, most extensively discussed as electrical problems, and reference to such books as Maxwell's "Electricity" and Thomson and Tait's "Natural Philosophy" would be likely to reveal close parallels. An examination of the writings of some of the early optical writers, Hamilton for example, might also be profitable. The detailed proof of some propositions—for instance, the equivalence of a circular disc and a spherical cap with the same boundary—tended to obscure the more general result that the boundary to the surface, and not the surface itself, is the relevant factor. This is really the fundamental proposition in the photometry of surfaces, and it is surprising that though it has been known so long it is so easily overlooked. Mathematically, the result at once follows from the fact that the integral is of a type which may be transformed from a surface integral to a line integral round the boundary of the surface.

The AUTHOR (communicated) in reply to Mr. T. SMITH: I shall be very glad to have the references to the previous work on this subject if Mr. Smith will be so kind as to send them to me. I have been able to find nothing except the work of Hyde and Foote referred to in my Paper, and neither of these authors refers to any previous treatment of the matter, except a Paper by Bassett Jones before the Illuminating Engineering Society of New York, "Trans." 5, p. 281, 1910.

The only analogous problem of which I have been able to find a treatment is that of the attraction of a uniform circular disc for a point on its axis. (Thomson and Tait's "Natural Philosophy," 477, Vol. I.) The expression obtained is not the same in the gravitational problem, however, as in this the attraction depends only on the first power of the cosine of the angle between the normal and the line connecting the points considered, whereas in the illumination case it is the square of the cosine that is involved. This arises from the fact that the interposition of matter between two particles does not affect the force with which they attract each other. Thus, there is nothing in the gravitational problem to correspond with the cosine law in the illumination problem.

IX. *A Comparative Method of Testing Thermionic Valves for Passing No Reverse Current at High Voltages. By* N. W. McLachlan, *D.Sc., Eng., M.I.E.E.* (*From the National Physical Laboratory.*)

Communicated by F. E. Smith, O.B.E., F.R.S.

Received October 23, 1919.

In using two electrode thermionic valves for the determination of the maximum or peak voltage of a high-tension alternating supply, and in the determination of the decrement of a circuit in which the damping is very high, it is essential that the valve should pass no reverse current. In order that the above condition is fulfilled, it is necessary to carry out tests at high voltages. The voltages used by the author for experimental work were generally of the order of 10,000, but tests on the valves were conducted up to 18,000 volts.

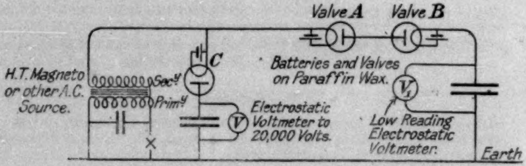

Valve A
Valve B
C
Batteries and Valves on Paraffin Wax.
H.T. Magneto or other A.C. Source.
Sec^y
Prim^y
V_1
Electrostatic Voltmeter to 20,000 Volts.
V
Low Reading Electrostatic Voltmeter.
X
Earth

Fig. 1.—Diagrammatic View of Apparatus used for Testing Valves for Passing Reverse Current. (Valve *C* and the Electrostatic Voltmeter reading to 20,000 volts, are used to measure the peak voltage applied to valves *A* and *B*. The negative pole of the battery for valve *C*, should be connected to the source of *A*, *C*. In the diagram the battery connections are shown reversed. When measuring high voltages this is of no importance since the drop in the filament is negligible.)

The simplest method of testing a valve for passing reverse current is to use a high-tension steady source. This can be obtained in the manner indicated by Prof. Fortescue in his recent Paper before the Physical Society.* It often happens, however, that the apparatus needed for this method of obtaining direct current is not available. In this case it is necessary to employ other means.

The apparatus used for testing the valves is shown diagrammatically in Fig. 1. The filament of valve *A* is connected to the high-tension side of an alternating-current supply. The anode of *A* is connected to that of *B*. The filament of *B* is connected to one plate of a condenser, which should have good insulation and dielectric properties. The other side of the condenser is put to earth or to the other side of the alternating-current supply. A low-reading electrostatic voltmeter (which should not leak) is connected across the condenser. Both of the batteries for the filaments of *A* and *B*, and also the valves themselves, should be supported on blocks of paraffin wax. In place of a high-tension transformer it was found more expedient to use a polar-inductor magneto which could be run up to 1,500 revs. per min. By cutting out the safety gap it was possible to obtain 20,000 volts. During the operation of the magneto, which would have given 100 sparks per second—*i.e.*, equivalent to a frequency of 50 ∼ per second—if used in the normal manner, the peak voltage is alternately positive and negative. On reference to Fig. 1 it will be seen that valve *A* is set to rectify the positive semi-alternation and valve *B* the negative semi-alternation. Thus, if neither of the valves passes reverse current, there will be no reading on the voltmeter *V*. If, however, one valve passes reverse current, whilst the other does not, there will be a reading on the voltmeter, since a current will flow into the condenser and charge it each time the valve passes current, until finally the difference between the peak voltage and the condenser voltage is such that reverse current just ceases to pass. It is impossible to discriminate, however, which of the two valves passes reverse current. This can only be done when either *A* or *B* is known to pass no reverse current at the highest voltage used during the test. Valves can now be obtained, two of which, when tested by the above method, pass no reverse current. The author recently tested two O.R.I. Osram valves up to 18,000 volts by the above method, and obtained no reading on the voltmeter, which latter would readily indicate 5 volts. The filament current of each was 5 amperes, and the duration of the test 20 minutes. Using one of these valves as a standard, it was possible to test other valves when the filament currents were varied. The reverse current depends to a certain extent on the bombardment to which the anode is subjected. If this is severe, the valve will pass reverse current; but under ordinary working conditions there is generally no necessity to resort to excessive bombardment.

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* "Proc." Physical Soc., Vol. XXXI., August 15, 1919.

Measurement of the Decrement of Highly Damped Oscillations with the Aid of a Thermionic Valve.

In measuring the decrement of highly-damped oscillations—*e.g.*, the oscillations which occur after the interruption of the primary circuit of a magneto—a wavemeter is sometimes of little assistance. In cases where it is possible to use a wavemeter, the following provides an additional method. The apparatus is arranged as shown in Fig. 2, where a tapping is taken off the inductance in the oscillation circuit, or a connection made across the condenser, according to the voltage which is available. The valve, which is known to pass no reverse current, is set to rectify, say, the negative portions of the oscillations. The condenser will therefore become charged

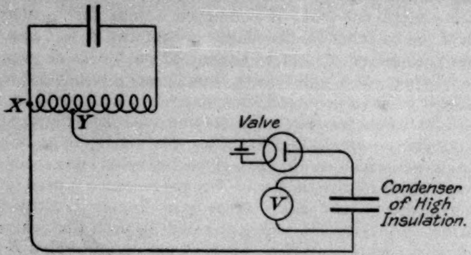

FIG. 2.—DIAGRAM OF APPARATUS USED IN MEASURING DECREMENT.

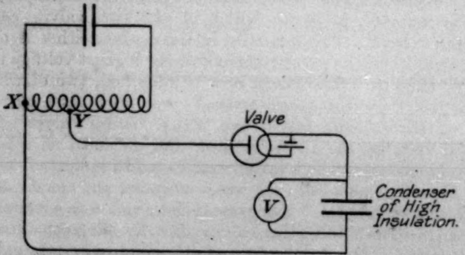

FIG. 3.—DIAGRAM OF APPARATUS USED IN MEASURING DECREMENT.

to a potential equal to the peak voltage which occurs between *X* and *Y* in Fig. 2. This potential is proportional to the peak or greatest positive voltage of the train of oscillations. In order to measure the greatest negative voltage, the valve is reversed as shown in Fig. 3. This can be done by means of a switch. The square of the ratio of the voltmeter readings is the decrement of the oscillations per period. It should be remarked that the condenser in parallel with the voltmeter might have some influence on the oscillations in the remainder of the circuit. This, however, is only transient, because once the condenser is fully charged no current flows in the valve circuit, except that due to leakage. The latter can be reduced to a very small quantity by observing the precautions already mentioned, and taking special care that the outside of the battery is dry.

Measurement of the Damping in a Magneto.

The foregoing method of measuring decrement can be applied to the magneto. The oscillations which occur in the primary and secondary circuits after the contacts separate are not damped sine waves, so that the term decrement does not strictly apply. In this case, therefore, the damping will be defined as the ratio of the peak value of the first positive semi-undulation to that of the first negative semi-undulation. It is useful to know this ratio, since it is a rough index of the effect of the iron loss.

There are two values of the damping which can be obtained from the magneto when a current is broken in the primary. One is found with the armature stationary, and fixed in the position at which break would occur in normal running. The other is found when the armature is rotated. Dealing with the case in which the armature is stationary, the circuit of Fig. 4 is utilised. This has been outlined in another communication, and a further description is unnecessary.* It is important, however, to point out that the direction of the current should be the same as that of the induced current during rotation. The reason for observing this precaution can be studied with the aid of hysteresis loops. Before the current is passed through the primary winding, the magnet produces a polarising flux in the armature core, its magnitude being greater at retard than at advance. The current, according to its direction, either increases or decreases this flux. The flux change in the armature, and therefore the peak voltage due to cessation of the current, depends on this.

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* "Proc." Physical Soc., Dec. 1919.

The reading of the voltmeter will be either the first positive or negative peak. It is preferable to obtain the former with the connections of Fig. 4, since there is then less chance of leakage. This can easily be managed by suitable choice of the rotor position. The negative peak can be secured in two ways: (*a*) Reversing the valve connections, (*b*) turning the rotor through 180 electrical degrees and reversing the current. In the latter mode of procedure it is essential to assume symmetry for the two positions of the rotor. The determination of the damping when the magneto is in operation can be effected by the same apparatus if the cam is arranged to provide only one break for a rotation of 360 electrical degrees—*i.e.*, alternate breaks are omitted. Under these circumstances, the history of the iron between successive breaks is different from that in normal running. The values of the peak voltages are due to interruption of the current plus the voltage due to rotation, but allowance can be made for the latter.* For strict comparison of this value of the damping with that derived with the armature at rest, the currents should be of equal magnitude.

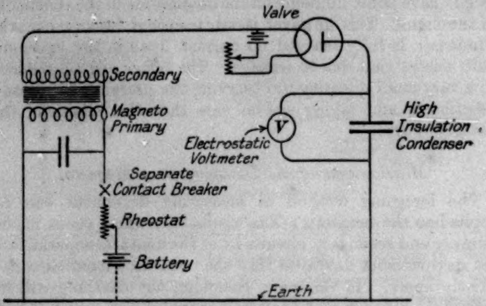

FIG. 4.

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* "Proc." Physical Soc., Dec. 1919.

ABSTRACT.

A simple method of testing diode valves for passing no reverse current at voltages up to 20,000 is described. The anode of a standard valve, known to pass no reverse current, is connected to that of the valve under test, and the filaments are joined to the terminals of a high-tension alternating voltage supply. A low-reading electrostatic voltmeter, with parallel condenser, is arranged to indicate if the valve under examination passes reverse current.

A description of a method of using a diode valve to measure the decrement of a highly damped oscillation is given. This consists in arranging the valve connections of the circuit so that the first positive peak is obtained. The valve connections are then reversed, and the first negative peak is found. This method is applied to the measurement of the highly damped oscillations which occur after the primary current is broken in a magneto.

X. *On Recording and Reproducing Sounds by Means of Light. By* A. O. RANKINE, *D.Sc., Professor of Physics in the Imperial College of Science and Technology.*

RECEIVED NOVEMBER 14, 1919.

(1) *Outline of Experiments.*

IN a recent communication to the Physical Society, I have described a method of transmitting speech by light.* In the discussion following the reading of the Paper it was remarked that the method could be adapted to recording sounds photographically, and afterwards reproducing them. Experiments made since then have shown that this is the case, and the results are interesting in several respects.

It has been customary hitherto to record sounds by transverse vibrations, such as, for example, the trace made by a vibrating spot of light on a moving photographic film. The record thus obtained does not lend itself readily to reproduction; but if, instead, sounds can be recorded photographically by means of changes of *intensity* of a beam of light, reproduction becomes quite easy. This process is not new, having been used in 1900 by Rühmer† in connection with experiments on the electric arc. It was developed by him as a means of testing the conditions under which the greatest variations of light intensity could be brought about by the action of microphonic currents superimposed on the current passing through the arc. He finally succeeded both in recording sounds by this means and in reproducing them; but he points out that the method requires laborious manipulation, owing, mainly, to the difficulty of keeping the arc in the sensitive condition. The photophone transmitter described in my previous Paper avoids this difficulty entirely, and enables photographic records of sounds to be obtained with great ease and precision. This is evident from the fact that the examples shown in Fig. 2 are the results of first attempts, and involved no very careful preparation. Probably they can be improved, but they are already sufficiently good to give accurate reproduction of the corresponding sounds.

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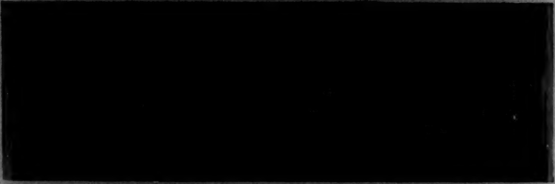
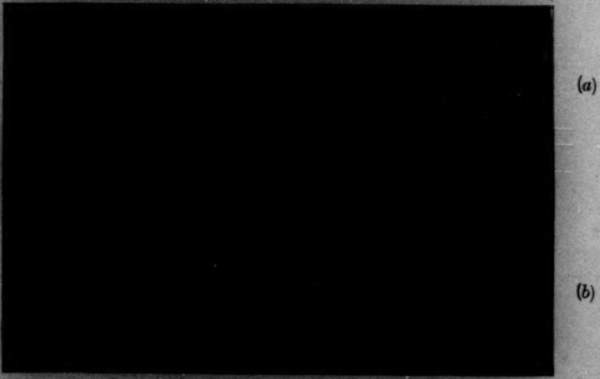
* "Proc." Physical Soc., Vol. XXXI., p. 242.

† Ernst Rühmer, "Wireless Telephony," 1907.

Example I.—Tuning Fork Frequency 512 per sec.



Example II.—Open Organ Pipe. Fundamental Frequency 520 per sec.



Example III (*a*).—Beginning of word "Five."
III (*b*).—Beginning of word "One."

Example IV.—Towards the End of the word "Five."

Fig. 2.

To face p. 79.]

(2) *Production of the Records.*

It is unnecessary to describe again the photophone transmitter. The light from a 500 c.p. Pointolite lamp was passed through the transmitter, and the emergent fluctuating beam, instead of being projected to a distance, was concentrated upon a narrow horizontal slit, S (Fig. 1), by means of the lens L_1. A second lens, L_2, was used to focus an image of the slit upon a band of kinematograph film, which could be moved in a vertical direction. The film was contained in a special camera, actuated by clockwork, which enabled about 2 metres of film to be run at nearly constant speed. In order to obtain as much detail as possible, the image of the slit on the film was made very narrow—between 0·1 mm. and 0·2 mm.—and the film run at considerable speed—about 170 cm. per second. In these circumstances the film was, perhaps, rather under-exposed, and improvements could, no doubt, be obtained by using a source of greater intrinsic brilliancy.

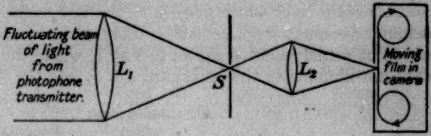


Fig. 1.

Examples of the results obtained are shown in Fig. 2, which represents positive prints taken from portions of the films. Example I. is the record of the vibrations of a tuning fork (frequency 512 per second) bowed and then held near the receiving trumpet of the transmitter. As might have been expected, the fundamental frequency only is indicated. Example II. was produced by an open organ pipe of fundamental frequency, 520 per second. Here the effect of the first overtone (frequency 1,040 per second) is seen superimposed on the fundamental. Examples III.*a* and III.*b* represent respectively the beginnings of the words "five" and "one." Space is not sufficient to show the complete words, each of which, although only a monosyllable, has, on the scale shown, a visible record about 40 cm. long. Example IV. is the record of that part of the word "five" where the "*i*" sound changes to the "*v*" sound. Progression is from left to right, and the change of character is shown fairly well. Sufficient photographs have been taken to show that in different records of the same word the general features are repeated, although the identity of detail is not, perhaps, as complete as might have been expected. It appears probable that even the same person, speaking the same word several times, does not actually repeat the vibrations identically. At any rate, it is totally unnecessary for him to do so, for, as will be seen later, speech communication requires only the existence of the principle features of the vibrations.

(3) *Reproduction of Sounds from the Records.*

In order to reproduce the sounds from the films, all that is necessary is to focus the light from a narrow source upon the film and allow the light which penetrates the film to fall upon a selenium cell connected with a battery and telephone receiver. A convenient arrangement is shown in Fig. 3, where the light from the arc illuminates the horizontal slit *S*, and the lens *L* produces an image of the slit on the film at *F*. The light penetrating the film fluctuates when the latter is moved rapidly, and sounds are heard in the telephone receiver. The reproduction thus obtained is good, and there is a quite remarkable amount of latitude in adjustment without noticeable deterioration of articulation. In the first place, it may be noticed that there is no need to use in reproduction a positive of the original negative film; the negative itself serves quite well. This means that the phase of the vibrations may be reversed throughout without any effect on the sounds heard. Secondly, it is apparently quite unnecessary to take any special care in the development of the film with a view to securing that the opacity is proportional to the intensity of the incident light. It is sufficient that the main frequency charac-

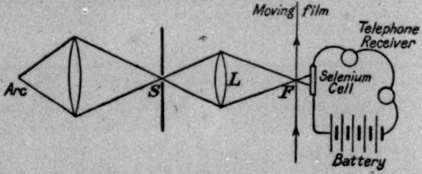


FIG. 3.

teristics should be represented by alternations of opacity. Further, it is not essential that the speed of the film during reproduction should be precisely the same as that during recording, in so far as the articulation of a word is concerned. The general pitch, of course, changes with speed; but it is possible to recognise the same word even if its record is run at a wide range of speeds. But the most remarkable fact of all is the small effect produced by alterations of width of the illuminating slit. The width of the slit image on the film during recording was less than 0·2 mm., yet in reproduction one can increase its width to considerably over 1 mm. without any marked effect on articulation. This means that the ear can be very easily deceived, and that only a rough approximation to the original vibrations is necessary in the reproduction of sounds required in ordinary listening. The practical effect is that there is no need for the film to be run at so great a speed as that used in these experiments. Probably a speed of 40 cm. per second would be quite sufficient with a slit image 0·2 mm. wide both in recording and reproduction. From the point of view of the visual examination of the records, higher speeds are, of course, preferable.

(4) *Significance of the Records.*

An interesting feature of the records is that they provide pictures of sound waves much more exact than is possible by representing them in the usual way as transverse vibrations. They give, in fact, a fair representation of the condition of *longitudinal* displacements which travel through the air. The changes of photographic density on the film record correspond to the changes of density in the air, and the record of any sound may thus be regarded as an instantaneous picture of the succession of compressions and rarefactions in the associated sound wave.

Different words have records which differ greatly in appearance. By increasing further the speed of the recording film in relation to the width of the slit, it is probable that these differences can be still further accentuated, so as to show clearly the differences between such sounds as, for example, "th" and "z." In this respect they may be useful to students of phonetics. It might even be possible for persons to overcome the difficulty of saying "th" by learning to create the pressure changes indicated on the "th" record.

In conclusion, I wish to express my best thanks to Dr. A. B. Wood for the loan of the camera used in these experiments, and to Mr. F. C. Toy for kindly assisting me in making the records.

ABSTRACT.

The light from the photophone transmitter described by the author in a previous communication (Vol. XXXI., p. 242) is concentrated on a narrow slit, an image of which is produced by means of an auxiliary lens on a strip of cinematograph film, the slit image being transverse to the film. The variation in intensity of the light when sound waves are received by the instrument are recorded as variations in the density of the film after development. For reproduction of the sounds, light from an illuminated slit is focussed on the film, behind which a selenium cell, connected to a telephone circuit, is placed. As the film moves, the intensity of the light reaching the cell varies in accordance with the density of the film, and the corresponding sound is heard in the telephone.

DISCUSSION.

Prof. **Bragg** asked if the records contained any clue to the solution of the question how many vibrations were required to give the sensation of pitch. He thought students of phonetics would welcome the apparatus.

Mr. T. **Smith** pointed out that no advantage would accrue from the use of a cylindrical lens in the reproducing apparatus, unless the form of the selenium cell rendered it desirable to spread the light over a longer strip than could be obtained with a spherical lens.

Mr. W. C. S. **Phillips** asked if the experiments would throw any light on the question of resonance in the mouth. The fact that some people speak much more easily than others is generally attributed to the better production of sound. It is sometimes found that, of two voices, that which is loudest and clearest when heard close at hand may not carry nearly so well to a distance. Possibly records of this kind might throw some light on these questions.

Mr. J. **Guild** said he hoped the author would continue his experiments until he produced a proper gramophone. Modern gramophones had improved enormously in quality of tone and accuracy of reproduction ; but the way in which the time of musical items was juggled with in order to fit them into the record was very annoying. With film records there would be no necessity to spoil pieces in this way.

Dr. D. **Owen** raised the question of the accuracy of reproduction. He did not agree with the statement that the ear was easily deceived. In some things—speech for example—we are easily pleased, because we unconsciously attempt to make the best of it; but in music we are much more critical. He thought there should be more accurate reproduction by the author's instrument than by the purely mechanical gramophone ; but there would, nevertheless, be distortion of the wave form due to the want of proportionality between the intensity of the sound and the density of the photographic image. Had the author tried reproducing, for example, the tones of a violin ? From the scientific point of view, the method was novel, but he thought the records might not be easy to interpret correctly. There was nothing, for instance, to correspond to a change of phase. Again it was stated that the brightness of the light focussed on the film was proportional to the excess of pressure in the wave, and that, therefore, the blackening of the film gave approximately a picture of the rarefactions and condensations. The light was, however, proportional to the displacement in the wave front, and not to the pressure excess.

Dr. VINCENT thought the instrument would be of value in solving many acoustical problems. One of these was the question of the formation of vowel sounds. The essential points differentiating between various theories of vowel formation could be sifted out from these records by running them at different rates. The difficulty with the older mechanical reproducers was that variation of the speed, except to a very limited extent, introduced mechanical troubles, such as chattering.

Mr. F. E. SMITH said they must not forget the valuable work done in the examination of sounds by the oscillograph. We had also to remember that the telephone came between the selenium cell and the ear, and was an additional cause of distortion.

Prof. RANKINE, in reply, said he had no information on the question asked by Prof. Bragg. A gramophone on the lines of his instrument would be an expensive luxury owing to the cost of the film. With regard to Dr. Owen's points, the records do show change of phase, as the instrument is arranged so that zero displacement corresponds to half intensity in the transmitted light; a displacement in one direction increased the brightness, and a reverse displacement diminished it. As to the meaning of the records, he didn't think the displacement of the diaphragm of the receiver would be that which the air would have had in its absence. The displacement of the diaphragm was more likely to correspond to the pressure in the wave front.

As regards the oscillograph, the difficulty is that for most sounds it is very insensitive, and the addition of an amplifier may produce distortion. In his own instrument there were, of course, several causes of distortion—the receiving diaphragm, the selenium cell and the telephone, in addition to the photographic film, as pointed out by Dr. Owen.

XI. *On some Experiments in which Two Neighbouring maintained Oscillatory Circuits affect a Resonating Circuit. By*

J. H. Vincent, *M.A., D.Sc.*

Received December 29, 1919.

The Ideally Simplified Problem.

If two valve maintained circuits (Bench 1 and Bench 2, Fig. 1) are so far apart as to have no influence upon each other and one of them (Bench 2) is capable of having its frequency adjusted as desired, then the combined effect on a resonating circuit such as shown in Fig. 1 when this is assumed to be so distant from each of the maintained circuits as to eliminate any coupling is merely the sum of the effects of the maintained circuits taken separately.

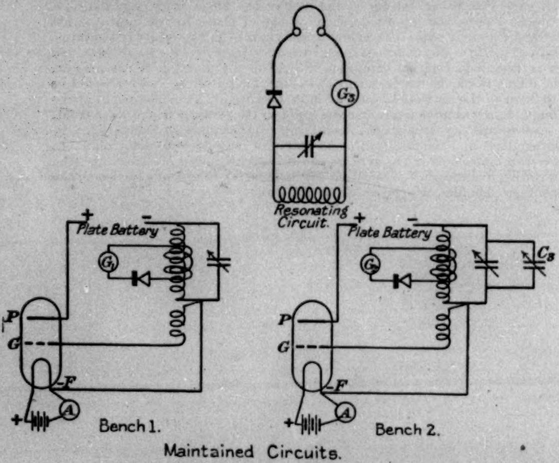

G_3
Resonating Circuit.
+
−
Plate Battery
G_1
P
G
F
+
A
Bench 1.
+
−
Plate Battery
G_2
C_3
P
G
F
+
A
Bench 2.
Maintained Circuits.

Fig. 1.

Thus when the Bench 2 circuit is tuned so as to be in absolute unison with the Bench 1 circuit the E.M.F.s and the currents in the resonating circuit will be the sums of the E.M.F.s and the currents due separately to the maintained circuits.

In particular the resultant current in the resonating circuit will be a function of the phase difference between the maintained circuits, and on this view this will be a mere matter of chance. If one induced current in the resonator were of the same amplitude but of opposite phase to the other then no resultant current would flow in the resonating circuit; while if the separate induced currents were of the same amplitude and phase then the combined current would be twice the magnitude of either.

The Actual Problem.

In these experiments the two-valve-maintained circuits are near enough to exert an appreciable effect on each other, which effect may be enhanced by a possible amplifying or minifying influence due to valves.

The resonating circuit (Fig. 1) had its condenser shunted with a carborundum crystal in contact with brass in series with a micro-ammeter (G_3) and a telephone. At times a vibrating galvanometer of low natural frequency, kindly lent to me by Mr. Jordan, was inserted in this circuit. The resonating circuit was not sufficiently distant for us to neglect the possible effects of coupling between this circuit and the maintained circuits; in consequence, a new coupling exerted on one maintained circuit by the other through the effect on the resonating circuit, may exist.

Finally, no direct measure of the current in the resonating circuit was made; the readings of the galvanometer G_3 in series with the detector were the subject of the observations so that the properties of the detector are involved in the work.

Apparatus.

The two maintained circuits were set up on two work benches placed a few feet apart as shown diagrammatically in Fig. 1.

There was no method available of measuring the alternating currents in the oscillators directly, but each main oscillating coil was wound with a few turns of insulated wire in series with a micro-ammeter and crystal detector. These micro-ammeters are referred to as G_1 and G_2 respectively.

The axes of the main oscillating coils of these two circuits were parallel and were connected up to the high potential batteries so that the currents through the valves made the end of Bench 1 coil nearer the resonating circuit have south-seeking polarity and the corresponding end of the Bench 2 coil have north-seeking polarity.

The axis of the coil of the resonating circuit was parallel to the line joining the ends of these two coils and a few feet away; so that if the two maintained circuits were in the same phase the induced E.M.F.s in the resonating circuit would be conspiring together.

The frequencies used in drawing the Figures 2, 3, and 4 were of the order of 100,000 per second, but widely differing frequencies have given similar effects.

Results.

As the frequency of the Bench 2 circuit is lowered by gradually increasing its capacity we pass from a condition in which Bench 2 is sharp on Bench 1 to that in which it is flat. We thus pass through the state of absolute unison. The capacity change per degree of small condenser in the Bench 2 circuit is about one-tenth of a centimetre, the main capacity being about 450cm. Thus a very slow adjustment of frequency is possible.

When the resonating circuit is carefully tuned to the Bench 1 circuit the circumstances are illustrated in Figure 2. When valve on Bench 1 only is alight the deflection of G_3, the galvanometer in the resonating circuit, is given by the line AB. When that on Bench 2 only is operating the deflection of G_3 is given by the curve CD, in which the ordinates are the deflections and the abscissæ the settings of the small condenser.

When both valves are lit the effects are shown by the curve EF. As the capacity increases the deflection G_3 increases; the pitch of the telephone note falls uniformly with the increase of the Bench 2 capacity as indicated by the line LM. Such lines were drawn from observations with standard forks by R. Koenig. As the point G is approached G_3 shows signs of vibrating. This vibration decreases in frequency until M is reached. The oscillatory current in the resonating circuit is practically zero, the deflection of G_3 is zero and the needle shows no signs of vibration. In the observations any direct-current galvanometer can be used. The description applied particularly to the behaviour of a Paul Uni-pivot instrument giving a deflection of 1 scale division for two micro-amperes. The two maintained circuits are now vibrating in unison and in the opposite phase.

On further increasing the capacity the two circuits are gradually brought into phase, at *J* this is complete. The change in Fig. 1 from *M* to *J* is brought about by an increase

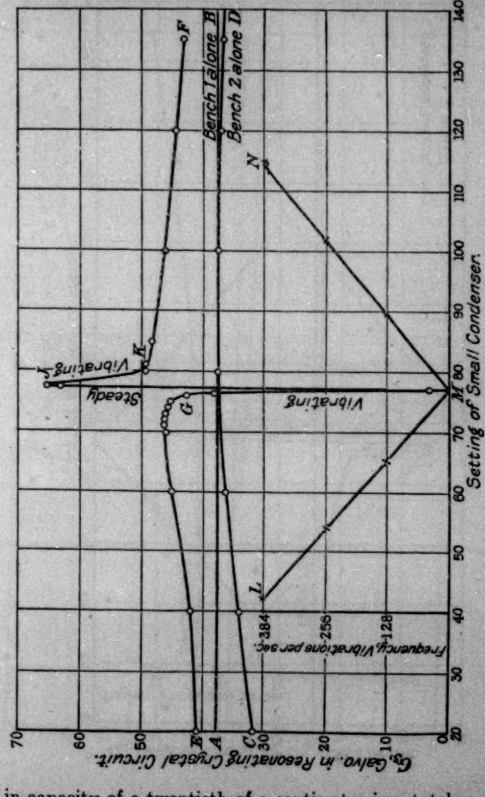


Fig. 2.

in capacity of a twentieth of a centimetre in a total capacity of 450cm. During this change from *M* to *J* the needle of G_3

does not vibrate; nor does a very low frequency galvanometer oscillate if included in the circuit.

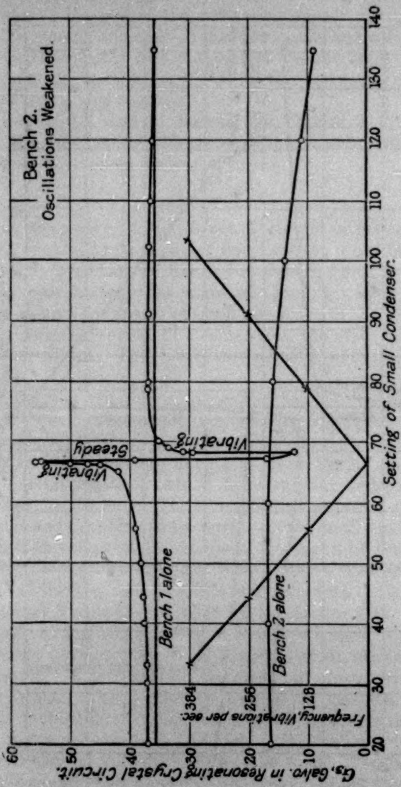


Fig. 3.

When the capacity is made still greater G_3 shows oscillations and the deflection falls

The line *NM* is again a practically straight line, the dead space from *M* to the projection of *J*, during which change in

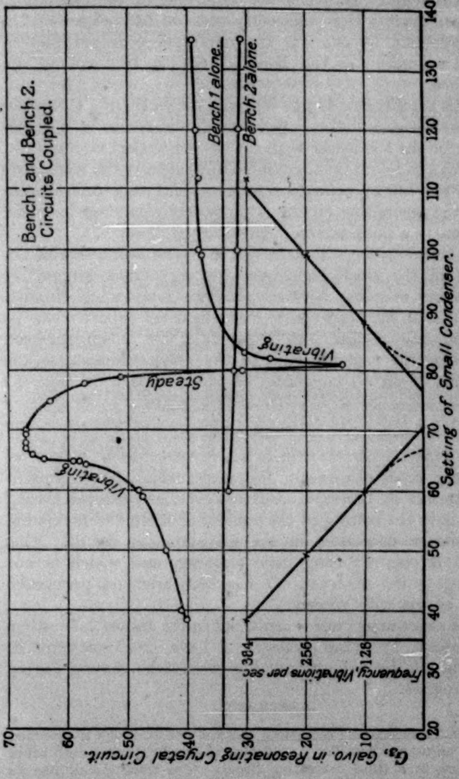


Fig. 4.

capacity does not alter the frequency, being too small to show clearly on records of the tuning fork observations.

The question immediately arises as to what settles whether the minimum or the maximum deflection of G_3 occurs first as the capacity increases.

It was found that when the amplitude of the oscillations in either circuit were sufficiently reduced (as judged by the galvanometers G_1 and G_2) the minimum occurred first or second according as the Bench 1 or Bench 2 circuit was weakened.

When a resistance is put into the Bench 2 circuit in series with the main oscillator coil the minimum occurs at a higher setting of the condenser than for the maximum. (Fig. 3.)

It was also found that accurate adjustment of the resonating circuit was not necessary. Similar results are obtained even when the resonating circuit is purposely mistuned in either direction to a considerable degree.

The curves giving the deflection of G_3 as a function of the setting of the small condenser are very much altered by introducing coupling between the two maintained circuits. In particular the dead space over which the capacity can be adjusted without change in frequency is very much increased by joining the negative ends of the valve filaments to the terminals of an air condenser of 12cm. capacity. (Fig. 4.)

Conclusion.

The experiments described above provide a means of observing small changes of frequency which have hitherto been studied by the method of beats. The observer merely has to note the setting of the condenser when the maximum or minimum deflection on the galvanometer occurs. This method is one requiring little training, and which is not fatiguing to the observer. It has been tried on previously known curves with success.

These experiments were carried out in Dr. Eccles' laboratory at Finsbury Technical College. I have great pleasure in thanking him for the encouraging interest he showed during their progress.

ABSTRACT.

Two valve-maintained oscillating circuits, one of which is provided with an adjustable subsidiary condenser of small capacity, each cause oscillations in a third resonating circuit. The third circuit has its condenser plates connected through a crystal detector and galvanometer. When the maintained circuits are in unison a very small change in the capacity of the subsidiary condenser causes the galvano-

meter needle to move through a large angle. The method provides a means of observing small changes of frequency previously studied by the method of beats.

DISCUSSION.

Dr. **Eccles** pointed out that Dr. Vincent's addition of the second high-frequency oscillation generator to the ordinary wave-meter method of working had resulted in an extraordinary enhancement of the accuracy of measurement of wave-length. It was very surprising to watch the sudden drop of the pointer of the table galvanometer in the crystal circuit as the second oscillator was brought into exact tune with the first. In this tuning process variations of capacitance of less than one-tenth of an electrostatic unit could be made evident by the table galvanometer in a total capacitance of 250 cm.—equivalent to detecting a change of radius of 1 mm. in a sphere of 25 metres radius. The method was far more accurate than that of the ordinary heterodyne wave-meter, and abolished the tediousness of using telephones. It promised to become a standard method of measuring wave-lengths.

Prof. O. W. **Richardson** said he was struck by the similarity of the curves to those representing anomalous dispersion in the neighbourhood of an absorption band. The explanation was doubtless similar in the two cases.

Mr. **Webb** said he had been working with the apparatus, and it was so sensitive that anyone moving even at a considerable distance caused large movements of the indicator. Despite this the method gave constant results for periods extending over several hours.

Prof. **Lees** asked if it mattered whether the condenser C_3 was in front of or behind the oscillating coil.

Dr. **Vincent** thought Prof. Richardson's simile was very suggestive. He had not been able in the present Paper to go into the detailed theory of the experiments, but he thought the theory when fully investigated would confirm Prof. Richardson's view. It was necessary to get good results, of course, to use all the tricks of the trade to maintain constant conditions in the circuits. Dr. Eccles and he had discovered a simple expedient for maintaining the frequency of the oscillations constant over long periods. This was to adjust the filament current so as to secure the maximum wave-length for the circuit in which the valve is employed. He could not answer the President's question off-hand.

XII. *Measurement of the Chief Parameters of Triode Valves.*

By W. H. ECCLES.

RECEIVED JANUARY 5, 1920.

1. When a tube is about to be put into use we require to know one or two commonplace facts concerning it, such as the appropriate filament current and voltage, which are easily found out or can be supplied by the maker. But there are other properties of the tube—such as the voltage ratio under conditions of normal use—which usually have to be specially measured, and which are essential for the formation of an accurate forecast of the performance of the tube in particular applications; for instance, in an amplifier or in a receiving set. A large number of parameters have been suggested for expressing numerically the various properties of triodes, but not all of them are equally important nor are all of them independent—that is to say, some of them can be deduced from others. By careful selection it may in time become possible to convey by quite a small number of parameters practically all the information requisite in practice. We shall, in fact, omit from the present section a number of suggested parameters or "constants," which depend jointly on properties of a tube and of the circuit it happens to be placed in, and we shall confine our attention to parameters belonging purely to the tube. Mixed functions of the tube and its circuits can usually be deduced from the data of the circuits of the true parameters of the tube itself.

2. At any given filament voltage or current the principal parameters of a triode are :—

(1) The co-ordinates of the mid-point of the straight part of the lumped characteristic.
(2) The voltage factor g.
(3) The differential co-efficient di_a/de_a or h_a.
(4) The differential co-efficient di_a/de_g or h_g.

Here i_a represents the anode current, e_a the anode voltage and e_g the grid voltage relative to the filament; the quantity e_a+ge_g is called the lumped voltage e_l and the curve connecting the anode current with the lumped voltage is called the lumped characteristic.

If these parameters are all supposed to relate to the regular parts of the lumped characteristic and not to extreme and unusual adjustments, the last-named parameter is not independent of the preceding two since $h_g = gh_a$. Other data of importance (but not discussed in this Paper) are :—

(5) The value of the anode voltage which just fails to draw the full saturation current given by higher voltages.

(6) The differential co-efficient di_g/de_a or g_a.

(7) The differential co-efficient di_g/de_g or g_g.

(8) di_a/de_f or h_f and di_g/de_f or g_f.

In addition, data concerning the knees of the characteristic curve are of value in connection with the use of the tube as a detector.

3. All these parameters belong wholly to the tube,—that is to say, they are not affected in value by the presence of the apparatus with which the tube is used and are, so far as is investigated, independent of the frequency. From a knowledge of these magnitudes the main aspects of the behaviour of the triode when connected to any kind of apparatus can be calculated. All of them can be deduced from the characteristic curves of a tube, but as the drawing of a number of complete curves is laborious we proceed to indicate various methods of determining the most important parameters direct.

4. The quantities g, h_a and h_g as well as some of the others are either mere ratios or are of the nature of conductances. Their measurement can, therefore, be reduced to the comparison of resistances, which can by the aid of resistance boxes be accomplished conveniently and accurately. But several of the measurements, notably those of parameters defined by differential co-efficients, are concerned with small changes in relatively large currents or voltages, and it is evident that if all of a large current is passed through a galvanometer the sensitiveness for small changes must be low. We therefore adapt some of the general methods of the physical laboratory to our purpose, methods such as the various resistance bridges, the compensation method, or methods using the differential galvanometer ; in all of which the galvanometer can be maintained in its most sensitive adjustment and be kept near its zero, although large currents are running in the things being measured. Only the compensation methods as developed by the writer are described in this Paper.

Remarks on Direct-Current and Alternating-Current Methods.

5. Direct-current methods have certain advantages over methods using rapidly alternating currents and a telephone receiver, though they are not so quick and convenient. An alternating current gives rise to alternating electric and magnetic forces round the various portions of the circuit it flows in, and these forces excite alternating E.M.F.s in neighbouring conductors and also in the different parts of its own circuit; in other words, the accidental electric coupling and magnetic coupling between portions of the same and other circuits produce E.M.F.s and these add subtly to the E.M.F.s applied for the purpose of the measurement. The effects are greater the higher the frequency of the alternations, and are usually too easily evident at ordinary acoustic frequencies. If these unwanted E.M.F.s happen to be in phase with the applied E.M.F.s it is possible to obtain silence in the telephone, but the balance adjustment so obtained will usually be farther wrong the stronger the induced stray effects. If, on the other hand, the stray E.M.F.s are not in phase with the applied E.M.F.s it is impossible to obtain a perfect balance.

In order to minimise these alternating-current disturbances it is well to run as much as possible of the connecting wiring through metal pipes, or to wind the insulated wires with lead or tin foil, and to connect all this metal covering to earth. It is necessary also to place coils in such relative positions that they have the least possible coupling with other coils, and to place condensers and other conductors of large surface well away from each other.

Co-ordinates of the Mid-Point.

6. The first important parameters are the co-ordinates of the mid-point of the straight part of the lumped characteristic. The most obvious and direct method for determining the lumped voltage of the corresponding current at the mid-point of the characteristic curve is to apply known grid voltage and anode voltage, to increase the latter until the saturation current is reached, to divide the magnitude of this by two, and then reduce either grid or anode voltage till the current becomes half of the saturation value. But the last step demands the use of calibrated sources of voltage, and as these are not always at hand, the following less direct method is recommended.

The connections are shown in Fig. 1. The galvanometer I ought to be of low resistance compared with that offered to the current by the internal path between anode and cathode. The filament E.M.F. or current must first be set at the desired stationary value by aid of the voltmeter V or otherwise, and the variable resistance R made zero. Then increase the anode voltage E by steps till the current as registered by the galvanometer I ceases to increase. Observe these values of anode voltage and current and then add resistance R to the anode circuit until the galvanometer reading is reduced to one-half the saturation value or i_1. Then the required coordinates are i_1 and $e_1=E-Ri_1$. It should be noticed that this simple experiment gives information of great utility in connection with trials for amplifiers, for the voltage e_1 is that lumped voltage which must be applied to the tube in order to bring the steepest part of the characteristic into use; and if the voltage on the grid is zero, then e_1 is the voltage which must be applied to the anode. As an example, in a certain tube with a filament voltage of 5·2V a voltage of 300 secured a saturation current of 10 mA. By adding $R=2800\,\Omega$ in series the current was reduced to 5 mA, the grid being connected all the time to the negative side of the filament. Therefore, $e_1=300-28\times5=160$V. The result is, it should be noticed, a lumped voltage; it may be made up by any anode voltage and grid voltage that satisfy the equation

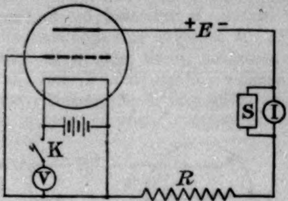


FIG. 1.

$$e_a+ge_g=160.$$

For example, if $g=10$,

we may have

$e_a=100$ and	$e_g=6$
130	3
160	0
190	−3
220	−6

Any of these settings will give practically the steepest part of the characteristic.

Compensation Method for Determination of h_a.

7. The method to be described is applicable at any setting of anode and grid voltages. The connections are explained by Fig. 2.

The anode battery is indicated by E, the grid potential divider by PD. A sensitive galvanometer with its shunt at IS. The compensation circuit includes the galvanometer, the battery e of smaller voltage than E, and the adjustable resistances R, R_1. The grid is permanently connected to the filament throughout the experiment and the grid voltage is, therefore, invariable. After the values of the anode, grid and filament voltages have been settled, the procedure is as follows: First, the galvanometer being shunted and the key K being closed, the resistance R_2 is given any value, R a small value (between 10 and 100 ohms in the case of a receiving tube) and R_1 a larger value (about 1,000 ohms), and the latter adjusted until the deflection of the galvanometer is zero. Then the key K is opened and R adjusted until the deflection is again zero. The shunt is now weakened, the key closed, and R_1 more accurately adjusted than before to obtain zero deflection, and then the key is opened and R improved in adjustment. The process is repeated to any required degree of accuracy.

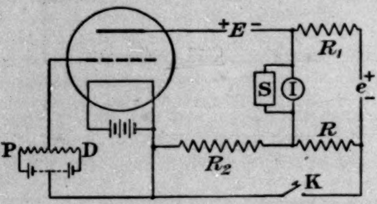


Fig. 2.

Zero deflection of the galvanometer implies that the P.D. between its terminals is zero. Therefore, when the key is open, the voltage applied in the anode circuit is $E-R_2i_a$; similarly, when the key is closed the voltage is E. Here i_a represents the anode current when the key is open, and is equal to the current in the compensation circuit. Let e represent the voltage of the battery in this circuit, then

$$i_a=\frac{e}{R_1+R}.$$

The current when the key is closed is e/R_1. Hence, on closing the key the anode voltage increases by the amount $R_2 i_a$, and the anode current increases by

$$\frac{eR}{R_1(R_1+R)} \text{ or } \frac{Ri_a}{R_1}.$$

Since the grid voltage is constant throughout the experiment, the ratio of the increase of anode current to the increase of anode voltage gives

$$h_a=\frac{R}{R_1R_2}.$$

A telephone and interrupter may be substituted for the galvanometer in the above method, the procedure being precisely as just described. See §5 regarding experimental precautions.

The following measurements were made by students at the City and Guilds Technical College, using direct current. They found that when R_2 is taken of small magnitude R must be finely subdivided in order to obtain accuracy. In the case below with $R_2=10$ ohms the subdivision of the resistance box used to supply R was not fine enough.

R_2.	R_1.	R.	h_a.
10	930·7	0·7	$7{\cdot}5\times10^{-5}$
200	928·6	13·0	7·00
800	927·5	52·0	6·88
1,000	926·0	65·5	7·1
1,045	924·0	67·0	6·9
1,200	927·2	78·0	7·03
2,000	927·9	130·1	7·06
5,000	928·9	323·5	6·95

Average value of $h_a=7\times10^{-5}$

$E=174$ volts.

$e=8$ volts.

A measurement made by Mr. R. H. Rinaldi on another triode gave

$$R_2=2050, \qquad R_1=903, \qquad R=43,$$

all in ohms, yielding $h_a=2{\cdot}33\times10^{-5}$mho.

Compensation Method for Determination of h_g.

8. The method to be described is applicable to any setting of the grid and anode voltages, and is explained by Fig. 3. The grid is connected to the filament direct when the key K is depressed and through R_3 when the key is open. After the values of the grid, anode and filament voltages have been settled the measurement is performed as follows :—

The galvanometer being shunted and K open, the resistance R is given a small value and R_1 a larger value, which is adjusted till the deflection of the galvanometer is zero. The key K is now depressed and R_3 adjusted by trial till the deflection of the galvanometer is again zero. The shunt is then weakened, the key opened and R_1 adjusted more accurately than before. The key is depressed and R_3 again adjusted. The process is repeated to any required degree of accuracy.

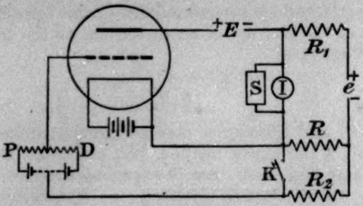


FIG. 3.

When the key is down the current in the compensation circuit and therefore that entering the anode is determined by R_1 in series with R and R_3 in parallel ; when the key is up this current is determined by R and R_1 in series. In the former case the current being equal to e divided by the resistance, is

$$\frac{e(R_3+R)}{R_1R_3+R_1R+R_3R},$$

and in the latter case is

$$\frac{e}{R_1+R}.$$

Hence, the increase of current when K is depressed, being the difference of the last two expressions, is

$$\frac{R^2e}{(R_1+R)(R_1R_3+R_1R+R_3R)}.$$

Again, when the key is down the voltage on the grid is that applied by PD; when K is up the grid voltage becomes smaller by the amount of the potential drop along R, namely,

$$\frac{Re}{R_1+R}.$$

Moreover, the anode voltage is the same whether the key is up or down, since zero deflection of the galvanometer implies that the P.D. between its terminals is zero, and therefore that E is the only voltage in the anode circuit. Hence, when the key goes from the open to the closed position the anode current and the grid voltage both rise, while the anode voltage remains constant, and therefore the ratio di_a/de_g is

$$h_g=\frac{R}{R_1R_3+R_1R+R_3R}$$
$$=\frac{1}{R_1+R_3+(R_1R_3/R)}.$$

A telephone and interrupter may be substituted for the galvanometer in the above method, the procedure being precisely as just described. See §5 regarding experimental precautions.

A measurement made by Mr. R. H. Rinaldi, using the direct-current method, gave

$R_3=90,\qquad R_1=907,\qquad R=43$ohms

and $\quad h_g=3{\cdot}46\times10^{-4}$mho.

Compensation Method for the Determination of g.

9. In regular parts of the characteristics the voltage factor g is equal to the ratio of h_g to h_a, and therefore need not be determined separately if h_a and h_g have already been measured. In any case g may be determined direct by the method sketched in Fig. 4.

The apparatus is the same as in the preceding paragraphs, except for the switch.

First, the compensation circuit is adjusted by variations of R_1, the switch being in the position marked 1; then the switch is moved over and R_2 is adjusted till the deflection is again zero.

When the switch is in the position 2 the anode voltage is $E-R_2i_a$, and the grid voltage is that applied by the potential divider PD. When the switch is moved to position 1 and zero deflection is obtained by varying R_2, the anode current is the same as before because made equal to the compensation current, which is unaltered. Moreover in position 1 the anode voltage is E and the grid voltage is Ri_a below that applied by PD. Thus the fall of the grid voltage Ri_a neutralises the rise R_2i_a of the anode voltage and therefore

$$gRi_a = R_2i_a$$

or
$$g = R_2/R.$$

When the grid current is extremely great, relatively speaking, an error arises in the above measurement because current is drawn from the compensation battery to the grid. This is, however, a rare event.

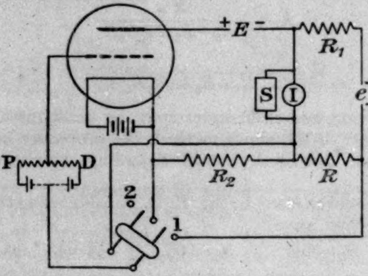


FIG. 4.

Measurements of g by the direct-current method give pairs of values for R and R_2, such as 50 and 900, 150 and 2,500, 220 and 3,960, all yielding 18 as the value of the voltage factor for the small receiving triode under measurement.

Combined Apparatus for Determination of h_a, h_g and g.

10. The three methods of measurement just described may be carried out by the aid of the single piece of apparatus sketched in Fig. 5, wherein the letters have the same meaning as in Figs. 2 and 3. The special feature is the six-point switch with which is combined the key K of Figs. 2 and 3. Mercury pools are used in order to ensure good contacts.

Referring to the circuit diagram, when the switch is in the left-hand position marked *a* the connections are as required for determining h_a, as described in §7, balance of the galvanometer being sought for in both the up and down positions of the chopper key. When the switch is in the right-hand position marked *g*, the connections are those of Fig. 3, and h_g may be determined by balancing the galvanometer for both the up and down positions of the chopper key. But when the magnitude of R_2 is so chosen that zero deflection is obtained when the switch is in either position with the chopper key up all the time, then the value of *g* is given by the ratio of R_2 to R, as explained in §9. In cases where the grid current is large, it is advisable to make R_3 zero during the determination of *g*.

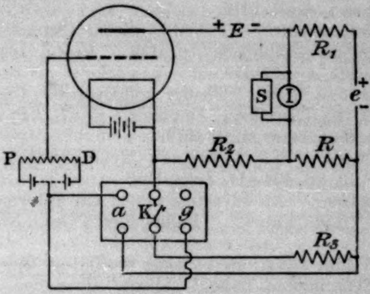


Fig. 5.

Appendix.

The following Papers deal with portions of the theory and describe methods of measurement bearing upon the points raised in the present Paper :—

T. Langmuir. "Pure Electron Discharge," "Proc." Inst. Radio Engineers, page 261, Vol. III.

M. Latour. "Discussion of the Audion," pp. 280-282, "The Electrician," Dec. 1, 1916.

G. Vallauri. "Comparative Experiments on Audions," "Elettrotecnica," pp. 335-342, June 23; and "Elettrotecnica," pp. 350-355, July 5, 1917.

J. M. Miller. "A Dynamic Method for Determining the Characteristics of Three-Electrode Vacuum Tubes," "Proc." Inst. Radio Engineers, p. 141, June, 1918.

E. V. Appleton. "A Thermionic Valve Slopemeter," "Wireless World," p. 458, Nov. 1918.

John R. Carson. "A Theoretical Study of the Three-Element Vacuum Tube," p. 187, "Proc." Inst. Radio Engineers, April, 1919.

Stuart Ballantine. "The Operational Characteristics of Thermionic Amplifiers," p. 129, "Proc." Inst. Radio Engineers, April, 1919.

H. J. Van der Bijl. "Theory and Operating Characteristics of the Thermionic Amplifier," p. 97, "Proc." Inst. Radio Engineers, April, 1919.

H. W. Nichols. "The Audion as a Circuit Element," "Phys. Rev." 13, pp. 404-414, June, 1919.

W. H. Eccles. "An Investigation of the Internal Action of a Triode Valve," "Radio Rev." Oct., Nov., Dec., 1919.

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ABSTRACT.

At any given filament voltage or current the principal parameters of a triode are:—

1. The co-ordinates of the mid-point of the straight part of the lumped characteristic.
2. The voltage factor g.
3. The differential coefficient di_a/de_a or h_a.
4. The differential coefficient di_a/de_g or h_g.

Methods of determining these quantities are described.

DISCUSSION.

Prof. Fortescue said the author had described some very nice methods for getting certain parameters of a valve; but he felt they were somewhat academic. There were two aspects in all these problems, namely, the practical and the instructional, and he thought Dr. Eccles had allowed the latter to predominate in devising the methods. For practical purposes one would frequently want to test the complete characteristic of a valve, and he would, on the whole, prefer the old arrangements. He did not think the author had emphasised sufficiently the importance of maintaining the filament current at a pre-determined value. The methods which he mentioned would give great accuracy, but they would be sensitive to very small variations of the filament current.

Prof. G. W. O. Howe said that the suggestion of a single "lumped" characteristic was attractive, yet in practical work it was very desirable to have the various characteristics for different grid voltages. He had been rather mystified by the author's opening remarks concerning Ohm's law.

To say that a valve obeyed Ohm's law when all the "constants" concerned follow complicted laws for different voltages and currents appeared to be stretching the meaning of Ohm's law to an inadmissible extent.

Prof. O. W. **Richardson** emphasised the remarks of the previous speaker regarding Ohm's law. He could see no reason for saying that Ohm's law applied where the resistance varied with the current and voltage.

Mr. F. E. **Smith** thought the methods given by Dr. Eccles would be exceedingly valuable where large numbers of valves had to be tested. He would like to emphasise Prof. Fortescue's caution about the constancy of the filament currents.

Prof. A. O. **Rankine** said: There is a point in connection with triodes concerning which I wish to ask the advice of the author. It refers to the use of triodes in that extreme case of low frequency, namely, when the voltage to be amplified is constant. Suppose that this E.M.F. is associated with a circuit which can be made of very low resistance, such, for example, a thermo-electric E.M.F. In that case, in the absence of means of amplification, the most sensitive arrangement is that in which the resistance of the measuring galvanometer is as low as may be consistent with galvanometric efficiency. The question is whether, in these circumstances, amplification of voltage by one or more triodes will be actually advantageous. If the limitation which I suspect does exist, it is important that the fact should be known; if it does not, the knowledge is, perhaps, of even greater importance, in view of the wide additional field which would be opened for the application of triodes. But, in either case, an authoritative opinion is required. The view I take of the question, subject to correction, is as follows: The output voltage of a triode is necessarily in a circuit of high resistance, for the latter includes the space between filament and plate, and it is therefore measured most efficiently by a detector of high resistance or impedance. But when the input voltage *need* only be in a circuit of *low* resistance, the fact that the E.M.F. is amplified by the triode may not be sufficient to compensate for the original loss of power involved by connecting the input voltage to the filament and grid, between which the resistance is large. A numerical example may be used to illustrate this point. Suppose that the E.M.F. to be amplified is E, and that it is associated with a resistance of 5 ohms. The most efficient galvanometer for direct measurement would have 5 ohms resistance. Thus, the current would be $E/10$ and the *power* operating on the galvanometer $\frac{E}{10} \cdot \frac{E}{2} = \frac{E^2}{20}$. Now, suppose the triode gives a voltage amplification of 10, and that the resistance between filament and plate is 50,000 ohms. In this case 50,000 ohms is the best resistance for the galvanometer, the current is $10E \times 10^{-5}$, and the power available $10^{-4} \cdot E \times 5E = \frac{E^2}{2{,}000}$, or only 1 per cent. of that obtained without the triode. Thus, several more triodes in cascade would probably be required to make up the leeway lost in the first, involving considerable increase of apparatus and multiplication of valve disturbances. The objection does not, of course, apply to the amplification of alternating and intermittent voltages, even when they occur in low resistance circuits, for in such cases the E.M.F. can be transformed up, without considerable less of energy, before being introduced into the triode.

Dr. **Eccles**, in reply, said that as regards the objections raised to mentioning Ohm's law in connection with the complicated actions in a triode vacuum tube, he thought it admissible to consider that Ohm's law applied instantaneously at any point of the curve connecting the space charge current with the E.M.F. applied between anode and cathode because the value of the current is obtained by dividing the sum of the E.M.F.s (including the back E.M.F. due to space charge) by the resistances (including the differential resistance). Both back E.M.F. and differential resistance are constant during an infinitesimal change of current and E.M.F., because they belong to the tangent line of the characteristic curve. In reply to Prof. Rankine, leaving aside rapidly varying currents and voltage, the use of a single triode valve for amplifying slowly varying currents or voltages is, as a rule, advantageous only when the supply of energy is strictly limited, and is in the form of a small quantity of electricity at relatively high potential or as a small current in a very high resistance. In such cases if the energy can be conveyed to the grid of the triode and spent in raising its potential. The release of energy in the anode circuit is very much greater and is in convenient form. In the case of the low-resistance thermo-junction problem suggested by Prof. Rankine the junction is a heat engine supposed provided with a source of heat capable of supplying energy for any length of time, and at a rate very largely in excess of what can be consumed in exciting the grid of the tube; therefore, the use of the triode can add neither to the convenience nor to the sensitiveness of the direct galvanometer method. Of course, if on any occasion a very high resistance galvanometer were the only one available it would be better to use a triode, or a sufficient number of triodes in parallel, with anode resistance approximately equal to that of the galvanometer.

XIII. *A Method of Measuring the Amplification of a Radio-frequency Amplifier.* *By* F. W. JORDAN, *B.Sc.*

RECEIVED NOVEMBER 29, 1919.

THE particular amplifier examined by this method was the French Radio-frequency Amplifier, Type L. 1. This instrument was designed for the reception of damped or variable radio-frequency waves such as occur in wireless signalling and telephony. This method of measuring the amplification was devised to suit this kind of reception. Continuous waves can also be received by combining a heterodyne with this amplifier.

The amplifier consists of six valves—three for the amplification of the radio-frequency, a fourth for the rectification of the latter and its conversion to audio-frequency, and the fifth and sixth for the amplification of the audio-frequency oscillations.

The radio-valves including the rectifying valve were coupled by inter-valve transformers of special design. The primary and secondary windings of each of these transformers consisted of a few turns of insulated wire on a laminated iron core of the shell type. The laminated iron sheets were very thin, so that the hysteresis and eddy current losses were small for currents of high frequency.

The audio-frequency valves including the rectifying valve were coupled by inter-valve transformers, the design of which is now generally well known. The listening telephone of 4,000 ohms supplied with this amplifier is usually inserted in the anode circuit of the last valve. The connection and disconnection of the receiver in this way is accompanied by a deafening click, which is very annoying during observations of feeble sounds. This was avoided by connecting the receiver to the terminals of the secondary of an equal ratio transformer—the primary forming part of the anode circuit of the last valve. This modification did not affect the amplification to any appreciable extent.

It was proposed to carry out this measurement in the space afforded by two rooms totalling about 30 ft. each way, and therefore special precautions had to be taken to extinguish all disturbing radiation and limit the received radiation to that which could be controlled by suitable apparatus.

The source of audio-frequency was a fork* alternator of novel design; which had been previously used for the measurement of audio-frequency amplifiers. The vibrations of the tuning fork of this alternator were maintained by a valve, and the circuit is shown in Fig. 1.

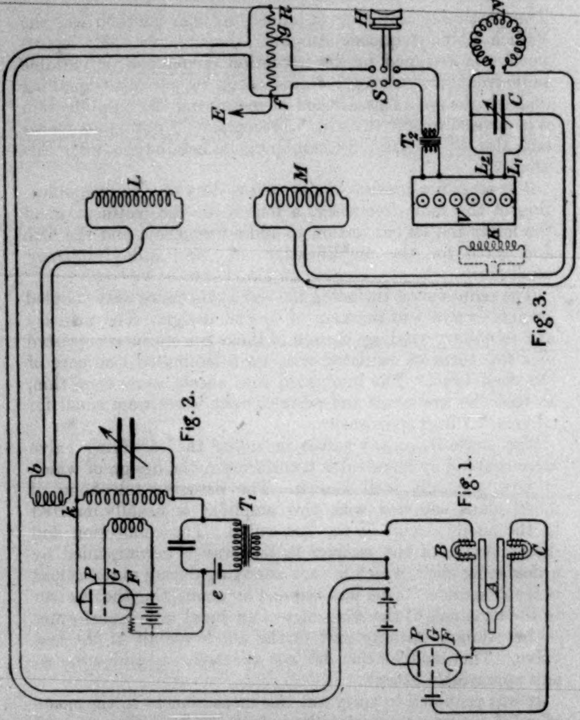

Fig. 1.

Fig. 2.

Fig. 3.

The Brown electro-magnets *B* and *C* were mounted close to the prongs of the tuning fork *A*. The coil of *B* was included in the plate circuit of the valve *P.G.F.* and the coil of *C* was connected in series with a dry cell to the grid *G*. The tuning fork can be set in vibration by closing the filament switch when the Brown electro-magnets have been properly adjusted. This arrangement provides a very steady source of sound and alternate current of known frequency for long periods, and the only attention required is the renewal of the charge of the filament battery.

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* Maintenance of the vibrations of a tuning fork," by W. H. Eccles, D.Sc., and F. W. Jordan, B.Sc., "The Electrician," June 20, 1919.

The purest tone from the tuning fork is obtained by adjusting the distance between the magnets and the prongs of the tuning fork to about 1 or 1·5 mm. The filament current is then adjusted to give the maximum intensity of sound, and then the desired A.C. in the plate circuit can be obtained by adjusting the voltage of the plate battery. The latter can be reduced to 8 volts without stilling the vibrations of the fork. The primary coil of the step-up transformer *T* (Fig. 2) and the 100 ohm. rheostat *R* (Fig. 3) are also included in the plate circuit of the valve.

The radiation from the circuit in Fig. 1 was very slight, and could not be detected with the amplifier at a distance of 20 ft.

A feeble source of radio-frequency waves for this measurement was afforded by the use of a Brown syntoniser (Fig. 2). The wave length of the oscillations was given on a circular scale attached to the variable condenser. The open coil of the oscillating fly-wheel circuit behaves as a radiator; but owing to the feebleness of the oscillations and the special precautions taken in this measurement, the direct radiation from this circuit could not be detected with the amplifier. A more suitable non-radiative feeble oscillator has since been designed for work of this character.

The voltage applied to the plate in this oscillator is that of the filament battery, and it was found with the aid of a slide potentiometer that the oscillations could be extinguished by reducing this voltage to about 3·8 volts. Since it was desirable to reduce the intensity of the free oscillations of this circuit to a minimum, a small accumulator *e* of 2·1 volts was connected in series with the secondary of *T*, and in opposition to the 6 volts of the filament battery. The intensity of the radio-frequency oscillations under these conditions in this circuit will be a function of the voltage at the terminals of the secondary coil of the transformer *T*, and vanishes when the total plate voltage falls below 3·8 volts. Other methods of superposing the audio on the radio-frequency oscillations were tried but not with the same success.

The maximum positive voltage applied by T to the plate circuit of Fig. 2 was determined by measuring how much e had to be increased to extinguish the fork note as received by the French amplifier. This was found to be of the order of 10 to 12 volts. Thus the voltage from T was sufficient to ensure that the radio oscillations were broken up into approximately semi-period trains of waves as shown in Fig. 4.

The audio radiation from an oscillator of this type ought to be inappreciable. This was verified by connecting a telephone to a coil which was closely coupled with that of the oscillator. An additional confirmation of this was made by connecting a loosely coupled coil l direct to the amplifier about 20 ft. away. The note of the fork was scarcely detectable until the coil l was tuned with a condenser to the radio-frequency oscillations, and then the sound was quite deafening.

The amplifier V (Fig. 3) was installed in a quiet room 18 ft. away from the circuit of Fig. 2. The receiving coil M was connected to the tuned circuit CN and to the input terminals L_1 L_2 of amplifier. The coil L was connected in series with l, which was coupled with the oscillating coil of Fig. 2. The input energy to the amplifier was controlled by adjusting the distance between L and M. The distance between the circuits of Figs. 1 and 2 was 4 ft., and the extreme distance between amplifier and circuit of Fig. 1 was 22 ft.

To be quite certain that no detectable radiation except the controlled portion reached the amplifier, (1) the axis of the oscillating coil of Fig. 2 was made vertical, while those of L and M and the radio coils of the amplifier were horizontal; (2) the leads from circuits of Figs. 1 and 2 to amplifier were encased in separate earthed lead sheaths; (3) the near point K of the oscillating coil of Fig. 2 was earthed; (4) other leads and circuits in the vicinity were removed, and when this could not be done were sheathed and earthed, and (5) the major part of the inductance N of the tuned receiving circuit CMN was a closed ring coil of a non-receptive character. Stray radiation could have been almost completely eliminated by enclosing the apparatus of Figs. 1 and 2 in earthed metal cases; but this would have given too much trouble. These precautions were effective in stopping stray radiation.

It was decided to select the fourth rectifying valve as a standard or unit with which to compare the amplifications of the various groups of other valves in this amplifier. The performance of such an unassisted valve does not differ very much from that of a crystal rectifier, and therefore its use as a practical standard is justified. The amount of energy transmitted to a valve or crystal rectifier depends to some extent on the electrical dimensions of the tuned receiving circuit. It was found that so long as the inductance and resistance of ringed coil *N* is kept within reasonable limits, the amplification is not much affected. Since a resonant receiving circuit is used in most methods of reception no account is taken in this measurement of the energy accumulated by resonance in such a circuit.

The procedure which was adopted in making a comparison of the different groups of valves was to vary the input energy to the receiving circuit *CMN* (Fig. 3) by means of the sliding coil inductance *LM* until the intensity of the fork note was equal to a comparison standard. A test with different observers showed that the range of adjustment of the sliding coils *LM* was equivalent to about 4 per cent. of the mean A.C. voltage applied to the receiving circuit. Thus with different groups of valves the relative voltage amplifications would be equal to the ratio of the mutual inductances of the coils *L* and *M* in the two cases—the energy amplification being equal to the square of this ratio.

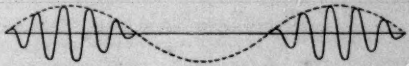


FIG. 4.

A comparison standard of sound of the frequency of the fork was given by inserting the dial potentiometer *R* (Fig. 3) in the plate circuit of the fork alternator. The potentiometer consisted of two non-inductive dial resistances, one of tens and the other of units. The leads from the circuit of Fig. 1 to *R* were encased in an earthed lead tube and the terminal *f* (Fig. 3) common to the telephone *H* and the rheostat *R* was earthed. The leads *m* about 1m. long were also encased in an earthed lead tube. Under these conditions no sound was audible in the telephone when the leads from this were connected through the small switch *s* to the same point *f* of the potentiometer *R*. The non-radiative character of this arrangement was tested by moving the contact *g* to the other end of *R* so that the full *P.D.* was applied to the leads *m* and the charge over switch *s* was placed in position where it connects the telephone *H* to the output terminals of the equal ratio transformer T_2. Under these conditions no sound was audible in the telephone, but if one hand was brought close to the fourth valve and the other stretched out towards *R* the fork note became audible. Reception of the latter kind was, of course, intentional, and does not occur in practice. The advantage of this as a comparison standard of sound is that a change of the fork alternating current will affect both the radio frequency and oscillations and this standard equally.

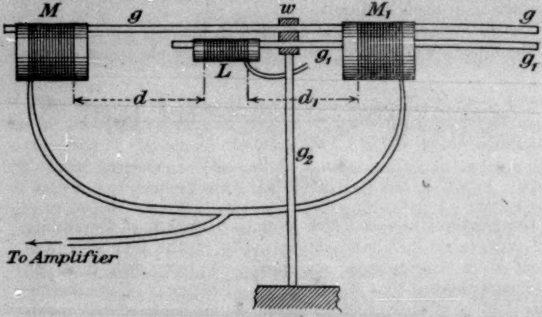


FIG. 5.

The method used for calibrating the mutual inductance between *L* and *M* presented some novel features, and is particularly applicable to this case where *L* and *M* are used for controlling the input energy to an amplifier.

The mutual inductance consisted of two coaxial coils *L* and *M* sliding on glass rods *gg* (Fig. 5), and the distance between them could be increased up to 130 cm. The advantage of this particular type of inductance is that the capacity effect between the coils is inconsiderable until the distance between them becomes small. The coil *L* consisted of 227 turns of No. 30 covered wire on a cylindrical tube of length 16·6 cm. and diameter 6·2 cm. The secondary *M* consisted of 66 turns of No. 20 wire on a tube 7·3 cm. long and diameter 10·6 cm. A block of wood *w* was paraffined and drilled longitudinally to receive the glass rods *gg* and $g_1 g_1$. The lateral distance between *g* and *g* was such that the coils *L* and *M* were co-axial when slung from the rods. The ends of the glass rods were supported by paraffined wooden clamps, and the rods were prevented from sagging at the centre by the glass rod strut g_2. Clips to grip the glass rods lightly were fitted to the sliding coils *L* and *M* to maintain the axes of the coils and glass rods parallel to each other. The coils were in this way isolated from surrounding objects, and secondary radiation and indirect transmission from *L* to *M* were eliminated. The mounting of the coils in this way does not permit of the highest accuracy being realised in the measurement of the mutual inductance, but in this work an error of 1 or 2 per cent. in the estimation or reproducibility of this quantity is not of much importance.

The capacity effect between *L* and *M* was tested by interposing an earthed screen of fine parallel wires between these coils. Only a very small change in the intensity of the fork note was noticed when the earth connection was removed from the screen unless the coil *M* was near enough to enclose a portion of *L*. The insignificance of the capacity effect is probably due to the fact that the radio-frequency oscillations were generated at a low voltage, and that one point *k* of circuit of Fig. 2 was earthed.

The mutual inductance between two coaxial coils is a function of the dimensions of the separate coils, their distance apart and the product of the numbers of turns in the coils. If the number of turns in one coil be halved and the other dimensions be unaltered, the mutual inductance will be halved. This gives the clue to the construction of a binary scale of mutual inductance. Another coil M_1 was constructed of half the number of turns of *M* by winding an equal tube with twin lengths of the same wire and connecting the lengths in parallel. Thus the dimensions of M_1 and *M* were the same, whilst the number of turns on M_1 was half that on *M*. The two coils *M* and M_1 were arranged on opposite sides of *L* and connected in opposition to the tuned circuit *CN* of the amplifier. The self inductance of the closed ring coil *N* was $0{\cdot}7\times10^{-3}$ henries and was much greater than that of *M*.

Suppose that it is desired to construct a binary scale of mutual inductance starting with M, say, at a distance of 1m. from L. M is placed at the latter distance and the position of M_1 is adjusted until the note of the fork is suppressed in the telephone H. This position of silence is well marked provided that the intensity of the radio-frequency oscillations is sufficiently great or the distance between L and M not too considerable. Let d and d_1 be the respective distances of M and M_1 under these conditions. Then the mutual inductance of L and M at distances d_1 is twice that at the distance d. M was next moved to distance d_1 from L and the adjustment of M_1 repeated. In this way a complete binary scale of Mutual Inductance between L and M was obtained and graphed for the purpose of this measurement. A ternary scale could be obtained in a similar way by making the turns of M_1 a third of those of M. A scale of this description is particularly applicable to amplifier measurement and is to be preferred to calculation of the mutual inductance as this takes no account of capacity effects and secondary radiation or indirect transmission, and in addition is extremely laborious.

If the coil L be made in two equal sections and these be traversed by the oscillatory current in opposite directions, then the mutual inductance between L and M will fall off still more sharply with increasing distance from L—approximately as the inverse of the fourth power of the distance between L and M. A radiating coil of this kind would be useful where amplifications of a higher order had to be measured.

Close coupling of M with L and ultimately with the coil of the oscillating circuit in Fig. 2 must be avoided. The existence of this can be easily detected by the use of the tuning condenser C in the receiving circuit since the reactionary effect of currents in M on L is to alter the wave length and sharpness of tuning and for very close coupling to set up oscillations of two different wave lengths. For this reason M was not brought too near to L, the inductance of L was made small in comparison with that of l and the coupling of l with the oscillating coil was made loose.

A final test for stray radio-frequency radiation was made in the following way: L was removed and the ends of the leads where they projected from the lead sheath were joined together. The receiving circuit CMN of amplifier had previously been tuned to the radio-oscillations. No sound was audible in the telephone until the coil *M* was moved to a position where it enclosed the exposed leads to *L*.

The amplifier was quite noisy at times owing to the presence of electrical machinery in the building and neighbouring factories; but there were periods when it was very quiet and measurements could be conducted quite safely.

The next step was to dissect the amplifier so that any arrangement of valves could be examined. The top covering plate of ebonite was raised and supported in the vertical position and the wooden back was turned down on its hinges. Nearly all the leads inside were disconnected and labelled. The leads where necessary were lengthened and the whole amplifier reassembled. The performance of the reassembled amplifier was found to differ but slightly from an unaltered *L*. 1. amplifier. Thus the rearrangement and slight lengthening of the leads had no deleterious effect on the amplification. The amplifier was supported on layers of felt to exclude mechanical vibrations and the observations were conducted in a quiet room.

The fourth rectifying valve, which has been selected as a unit, was isolated together with its associated transformers from the rest of the valves. The comparison sound was made feeble by connecting *H* (Fig. 3) through switch *s* to the first 2 ohm section of the potentiometer *R*. In the other position of the switch *s*, *H* could be connected through the transformer T_2 to the secondary of the inter-valve transformer of this valve. The primary coil of the radio inter-valve transformer was connected by the terminals of the tuning condenser *C* of the receiving circuit. The coil *M* was then moved to position where the fork note from the valve was equal to the comparison sound. The inter-valve transformers were associated with the valve in this measurement; but a slightly better result was obtained by the unassisted valve. The adjustment of *M* was made for wave lengths of 1,500 m. and 2,500 m. and also for different filament currents.

The fifth valve was then coupled with the fourth, and the adjustment repeated. The voltage amplification was then expressed as being equal to the ratio of the mutual inductances of the coils *L* and *M* in the two cases. The sixth valve was then coupled with the fourth and fifth. The amplification of the audio-frequency valves was found to be about eight to ten times that of the rectifying valve. The radio-frequency valves were examined in the same way, and the amplification by these was of the order of 60 to 70 times that of the rectifying valve. The amplification of the assembled amplifier was about 500 to 700 times that of the rectifying valve at the same filament current and for a wave length of about 2,000 m. These figures must be squared to obtain the energy amplification. The actual amplification increases enormously as the filament current is increased, and the figures above only refer to a comparison of the groups of valves with the rectifying valve at the same filament current.

It was noticed that there was a change of character of the note with different groups of valves and the sharpness of tuning with condenser *C* varied from group to group. The latter is easily accounted for by the fact that *N* and *M* were connected in parallel with the primary coil of the inter-valve radio-transformers, whilst the former is accounted for by distortion of the wave form by the iron cored transformers. The tuning was very sharp in the case of the completed amplifier. The amplifier was usually much quieter with the switch on left depressed to stud *R*, although this is not the best position for the amplification of short waves. Retroaction and auto-oscillation of this amplifier can, if necessary, be regulated by the use of a solenoidal coil *K* (Fig. 3) connected in series with *M* and placed at a little distance from the radio-transformers of the first three valves. Retroaction can be regulated by moving *K* to or from the transformers. For a certain critical position of *K* the amplifier becomes self-oscillatory, and for a slightly less intense retroaction the amplification is much improved.

The object of this work was not to attain a high order of accuracy, but to devise some trustworthy method of comparing the performance of the separate valves of an amplifier and of different amplifiers. Such results are very useful in guiding one in the proper selection of valves, and the correct design of the electrical equipment of an amplifier.

Finsbury Technical College.

ABSTRACT.

The audio-frequency oscillations were generated by means of a tuning fork, which was maintained in vibration by a triode valve. The alternating-current voltage from the plate circuit of this valve was applied through a transformer to the plate circuit of a feeble radio-frequency oscillator. The radio-frequency oscillations were in this way separated into alternate semi-period trains of waves. The input

energy to amplifier was controlled by a sliding coil inductance. The radiating coil of this was loosely coupled through an auxiliary coil with the oscillating coil of the radio-frequency generator and the receiving coil was connected to the tuned receiving circuit of the amplifier. The receiving coil of this inductance was adjusted so that the intensity of the rectified oscillations, as received through the amplifier by the telephone, was the same as that of a standard source of sound of the fork frequency. The latter standard was furnished by applying a small alternating-current voltage from the fork alternator circuit to the receiving telephone. The voltage amplification under these conditions is the ratio of the mutual inductances of the sliding coils (1) without the amplifier and (2) with the amplifier. A method of calibrating the mutual inductance and also methods of eliminating strong radiation are detailed in the Paper.

DISCUSSION.

Mr. F. E. SMITH congratulated Mr. Jordan on the results he had obtained. The difficulties of the method were undoubtedly considerable. At the National Physical Laboratory measurements of a similar nature had been made, and with similar apparatus except for the air core transformer, and the means for breaking up the train of undamped oscillations. There were, however, other objects in view than the measurement of amplification, and these had led to the introduction of a device for quickly changing the intensity of the oscillatory current. With regard to methods for producing a tonic train of undamped oscillations he had found some years ago that it was possible to do this by introducing two oscillatory circuits in a single anode circuit of a valve, one of these oscillatory circuits being of radio-frequency and the other of sonic frequency.

See also page 131 for comments by Mr. P. R. Coursey on this and the following Paper.

XIV. *On the Measurement of Amplification Given by Triode Amplifiers at Audible and at Radio Frequencies.* *By* F. E. Smith, *F.R.S., and the late* H. C. Napier, *M.A.* (*From the National Physical Laboratory*).

Received January 7, 1920.

1. *Measurement of the Amplification Given by Audible Frequency Amplifiers.*

Triode valves are used for a variety of purposes, but more especially as amplifiers of radio-frequency oscillations, as rectifiers of these oscillations and as amplifiers of the resulting groups of rectified oscillations, these groups being of audible frequency. In ordinary telephony, triode valves are desired to be amplifiers only, as any rectification leads to distortion.

The quantitative measurement of the amplifying powers of valves is a matter of considerable interest. In the early days of the war, with a three-valve set, voltage amplifications of many thousands and even of a million times were often spoken of, but never met with in our experience.

A set of four valves with their associated transformers is shown in Fig. 1. In many such sets the first valve is a rectifier and partial high-frequency amplifier, the other three valves being audible frequency amplifiers. Iron core transformers are then used between the first and second, the second and third and between the third and fourth valves. In the figure, the telephone is intended to be in series with the anode of the fourth valve at D_1D_2; but sometimes a transformer is interposed, the primary of which is in the anode circuit and the secondary is in circuit with the telephone. For amplification of audible frequencies only, an additional iron-core transformer is used between the source and the grid of the first valve. This is shown in the figure.

Measurements showed, as might be anticipated, that in an amplifier of this kind the transformers play an important part, and it was at times advantageous to substitute other transformers for those supplied with the set.

In amplifying sets for either audible or radio frequencies, but not both, the transformers between valves are similar, and, since an experimental receiving circuit is usually very dissimilar to the connecting circuits between the valves, amplification by a separate valve was first obtained in our experiments, and the intensity of the note produced in a telephone in the anode circuit of this valve was taken as the standard of reference. The valve so used and the transformer associated with it were similar to those in the set. In Fig. 1 this reference valve is *A*, and it is the amplification of the three valves *B*, *C* and *D* which is intended to be measured. The intensity of the standard note was usually adjusted at the source, so that signals could be just read with comfort.

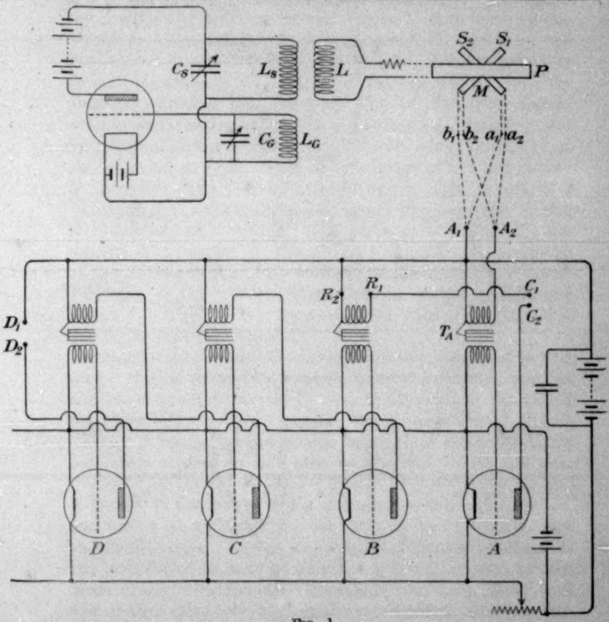


FIG. 1.

In some early experiments a buzzer was employed as a source of sound, but the noise and general disturbance due to it were very troublesome, and it was displaced by a triode valve set generating oscillations of an audible frequency. The anode oscillatory circuit consists of the inductance L_2 and the condenser C_2, these being adjustable and of such values that oscillatory currents of almost any audible frequency can be obtained. The great advantages over the buzzer are the constancy of the frequency and intensity of the oscillations, and the fact that no audible note is produced until a telephone circuit is in some way coupled to L_2.

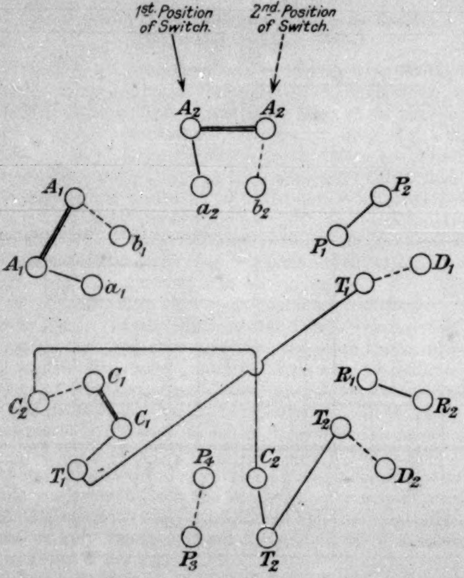

1st. Position of Switch.
2nd. Position of Switch.
A₂
A₂
a₂
b₂
A₁
b₁
A₁
a₁
P₂
P₁
T₁
D₁
C₁
C₂
C₁
R₁
R₂
T₂
T₁
P₄
C₂
D₂
P₃
T₂

FIG. 1B.

A coil, L, is loosely coupled to L_2, and the induced currents pass through the primary P of an air core transformer, M. This transformer has two secondaries, S_1 and S_2, crossed at right angles and capable of rotating together about a vertical axis. The mutual inductance between the primary and either secondary was measured for various angular positions of the latter by means of a Campbell inductometer. Either of the secondaries may be directly included in the grid circuit of the valve A or a transformer may be interposed. The anode circuit of A includes the primary of a suitable transformer, the secondary of which is in the grid circuit of the valve B. When the telephone (connected to T_1T_2, Fig. 1B) is included at C_1C_2 in the anode circuit of A, the primary of the transformer between the two valves is excluded from the circuit by a short-circuiting piece across R_1R_2. In these circumstances, suppose that the secondary S_1 of the transformer M is in circuit with the primary of the transformer TA; by means of a turning head (shown in Fig. 1B), S_1 is now disconnected, and the secondary S_2 is substituted, the telephone being simultaneously removed from the anode circuit of A and included at D_1D_2 in the anode circuit of the last valve; at the same time the short-circuit across R_1R_2 is removed. These changes are simultaneously made by a six-brush switch in a fraction of a second. The secondaries S_1 and S_2 are rotated with respect to the primary P until the intensities of the signals for the two positions of the switch are equal. The ratio of the mutual inductances between P and S_2 and P and S_1 is then a direct measure of the voltage amplification due to the valves B, C and D.

Amplifications up to 20 are conveniently measured in this way. If it is desired to measure greater amplifications, the number of turns on S_1 and S_2 may be different, or the switch may alter the number of turns in the primary P, the current through P remaining practically constant, or the value of the oscillatory current in L_3C_3 may be altered. In the last case the ratio of the mutual inductances between P and S_2 and P and S_1 is the one in which fine adjustments can be made, and has to be multiplied by the ratio of the currents to give the amplification. In this way a series of three selected valves has been found to have an amplifying power as great as 300. Most of our measurements have, however, been made with one or two valve sets, the usual magnification obtained with one valve being from 8 to 12 and with two valves from 20 to 80. The big variation in the latter case is due to variation in the valves employed. If two single valves, each of which will, when the currents and voltages are nicely adjusted, amplify a note 10 times, be used in cascade, the amplification obtained is in general less than 100, for it is usually necessary to adjust the filament current to avoid hissing in the valves. This trouble is, however, much smaller than it was a few years ago.

Fig. 1B shows the connections to a six-brush switch for making the changes referred to. The parts on Figs. 1 and 1B, which are indicated by similar letters, were connected together by leads. The telephone is connected to T_1 and T_2. A single brush connects two studs on a radius, and the angular movement of the switch is limited to the small angle between two pairs of neighbouring studs. When more than 20 volts was applied to the anode circuits, the telephone was connected to the secondary of a suitable transformer, the primary of which could be placed in an anode circuit by connecting it to the studs T_1 and T_2 of the switch.

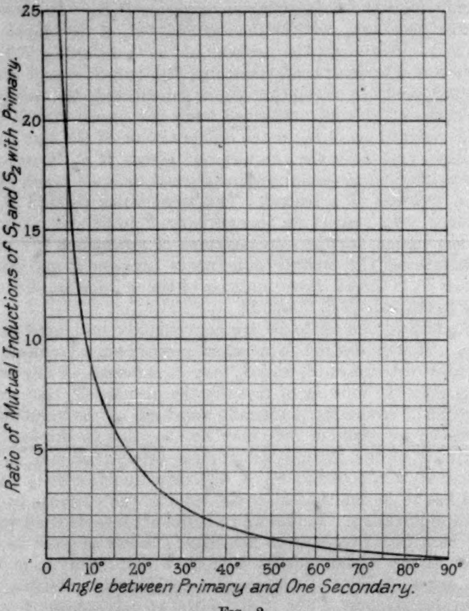


FIG. 2.

In making measurements the axes of the coils L_3, L_g and L were kept vertical and sufficiently far removed from the rest of the apparatus to prevent disturbing effects due to any direct action on the secondaries of the transformer M or any parts of the valve circuits. The axes of the coils of M were horizontal. Loops in leads were avoided, and when P was short-circuited there was a negligible amount of energy passing into the valve circuits from the telephonic frequency circuit.

Fig. 2 gives the ratios of the mutual inductances of the secondaries S_1 and S_2 with the primary P for angular displacements of the secondaries. For either coil the ratio is a maximum when the plane of the coil is in the plane of the primary.

It will be seen that if the amplification is greater than 20 the angle between one secondary and the primary will be less than 5 deg. The primary P was therefore divided into three parts, so that the number of turns in circuit with L could be varied, the ratio of turns used being 1 : 10 : 20. In a later apparatus the number of turns on a secondary can be varied.

Comparison with the Shunted Telephone Method.

Measurements were made to compare the results obtained by the method already described with those obtained by a shunted telephone method. The latter was a slight modification of the method usually employed, the energy of the current received in the telephone when two or more valves were used being cut down by a shunt until a balance was obtained against the energy of the current derived from one valve. The telephone was not shunted to secure silence. In this shunted telephone method the oscillatory energy in the grid circuit of the first valve was constant, whereas in the method previously described it was varied. If, under the conditions of the experiment, L_1=inductance of the telephone, R_1=resistance of the telephone, and R_2=resistance of shunt, then

$$\text{Current amplification} = \frac{\sqrt{(R_1+R_2)^2+L_1^2\omega^2}}{R_2}$$

In our experiments the resistances and inductances of the telephone and the shunt were measured at the frequencies employed in the amplification tests. In a particular case, when the frequency was 574 per second, the Sullivan telephone used had an inductance of 1·54 henries and a resistance of

3,900 ohms. The shunt resistance was of negligible self-inductance.

As examples, the following results are given :—

Valves.	First method.			Shunt method.	
	Angle of balance.	Ratio of turns in primary P, the current in P being practically constant.	Magnification	R_2 ohms	Magnification (approximately).
K_1 …………	8°	1	12	660	11
K_2 …………	$7\frac{1}{2}$°	1	$12\frac{1}{2}$	640	13
K_3 …………	10°	1	$9\frac{1}{2}$	740	10
K_3 …………	45°	10	10	700	10
K_1+K_2……	20°	10	45	185	37
$K_1+K_2+K_3$	6°	10	160	48	140
F_1 …………	12°	1	$7\frac{1}{2}$	920	8
F_2 …………	10°	1	9	850	$8\frac{1}{2}$
F_3 …………	10°	1	9	820	9
F_1+F_2 ……	$17\frac{1}{2}$°	10	50	170	40
F_2+F+F_3..	7°	20	260	33	200

There was some hissing produced when the three valves were used in cascade which rendered it necessary to slightly reduce the filament current.

For the shunted telephone method the number of turns in the primary P was kept constant, and the angular reading of the secondaries was also kept constant at 45 deg. To facilitate the readings, the six-brush switch was modified as shown in Fig. 3. The variable resistance R_2 was inserted between the studs marked T_1 and P_3, and P_4 and T_2 were connected together. R_2 was adjusted until equality of sound was obtained for both positions of the key. The reading of R_2 was all that was necessary for the shunt method. For the other method, the switches K_1 and K_2 were opened, and the secondaries of the transformer M were turned until equality of sound resulted. This gave the reading for the first method. Thus the comparison of the two methods could be made in a minute or thereabouts.

The two methods are distinctly different. In the first method the amplification is measured by *reducing* the incoming power, while in the second method the incoming power remains *constant*, the amplification being measured by reducing the effect of the amplifying action. If the alternating potentials applied to the valves are within the limits of the grid voltages which correspond with practically straight line portions of the anode characteristics, and if there is negligible grid current, the two methods are likely to agree, but not otherwise. This will be seen from a consideration of Fig. 4, which gives the change of anode and grid currents with change of grid voltage over a limited range. In the first method described in this Paper, if when one valve only is used the change of grid voltage on the valve is represented by AB, this will also be (for precisely similar valves) the change of grid voltage on the

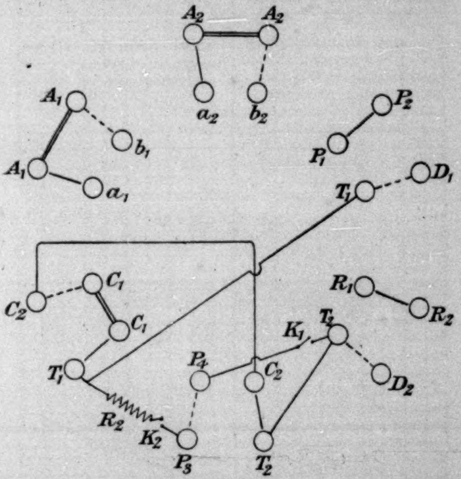


FIG 3.

last valve when a number are used, the change of oscillatory voltage on the first valve being reduced in order to achieve this result. In the shunted telephone method the oscillatory voltage on the first valve will still be AB, but that on the second valve will be much larger, and may extend over the bend of the characteristic, so that the ratio of the oscillatory grid voltage to the resultant change in anode current may change considerably from valve to valve. That this is so seems to be evident from the measurements, and since the purpose of an amplifier is not as a rule to make a great noise in the receiving telephone, but to amplify either weak or inaudible signals, so as to read messages with comfort, the shunted telephone method does not in general give the results desired.

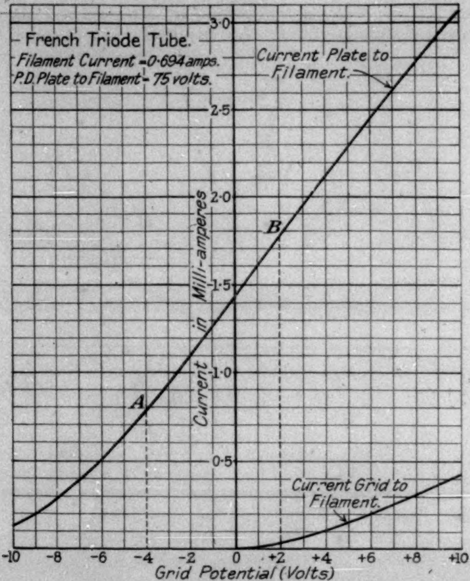


Fig. 4.

Fig. 5 gives the anode and grid characteristics of a Telefunken triode valve from a German submarine. It is rather soft, and good results were not obtained with such valves in cascade.

For valves to be satisfactory as telephone relays it is not only desirable for the rate of change of anode current with change of grid potential to be constant over the range worked over, but there should be negligible or constant grid current, or otherwise some rectification and therefore distortion results. This will be referred to in more detail in the next section.

2. *Measurement of the Amplification of a Radio-Frequency Amplifier.*

Radio-frequency amplifiers differ from those intended for the magnification of audible frequencies, the most conspicuous difference being in the inter-valve transformers, which are either without iron cores or have such in the form of very thin laminated sheets.

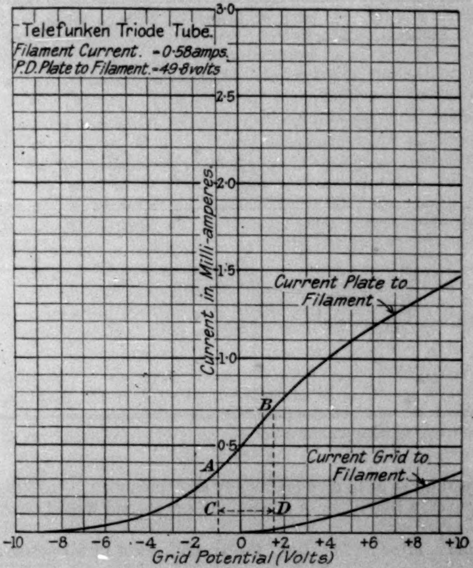


Fig. 5.

The method already described for measuring the amplification of an audible frequency set is not suited for one intended to amplify radio frequencies. The capacity of parts of the circuits with earth results in very irregular distribution of current, and only when extraordinary precautions are taken can the coupling between the coils of any air core transformer be regarded as a true measure of the coupling between the circuits. Measurements have been made by a method closely analogous to that for audible frequencies and with fair success; but a simpler method was devised, and this is now described.

The source of radio-frequency oscillations was a single valve, V (Fig. 6), with a tuned oscillatory circuit, L_1C_1, adjustable so as to give practically any frequency. In general, however, a frequency of one million per second was chosen for our experiments.

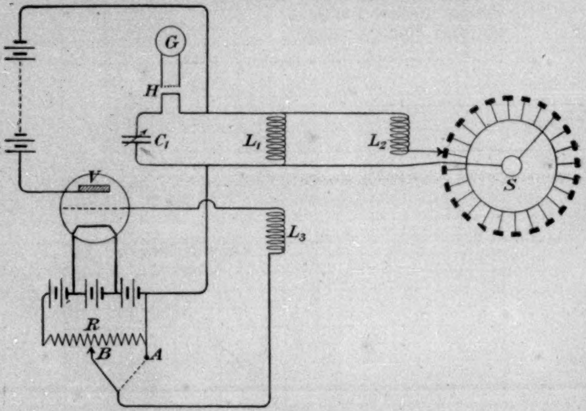


Fig. 6.

Since, after rectification, the amplification of a continuous train of waves of radio frequency produces no note of audible frequency in a receiving telephone, it is necessary either to break up the train into groups of audible frequency or to superpose some effect which will produce the desired result. It is easy to break up the train of waves at the source or at the receiving end, and it is equally easy at the source to superpose on the waves others of telephonic frequency, or at the receiving end to produce interference with a train of waves nearly equal in frequency (heterodyne). For our purpose, it appeared best to break up the train of waves at the source.

The train of oscillations was broken up into groups by throwing the anode oscillatory circuit out of tune at regular intervals corresponding to the period of the audible frequency desired. A rotating shaft *S* (Fig. 6) carried a solid brass disc, from which 24 studs projected into a layer of vulcanised fibre, so that their surface was flush with that of the fibre. A brush made continuous contact with the brass disc, and a second brush made intermittent contact with the studs. On making contact between a brush and a stud the inductance L_2 was placed in parallel with the inductance L_1 of the anode oscillatory circuit, thus throwing the circuit out of tune and stopping the oscillations. This method was adopted in preference to breaking either the anode or grid circuits, as in the latter case the oscillations did not always commence again on make, and an irregular note was given in the telephone of the receiving circuit. Moreover, if the leads carrying a direct current be broken, a further wireless disturbance is set up, as experiment abundantly verified. The plan adopted, however, left nothing to be desired. In the experiments the frequency thus obtained was about 800 per second.

The essence of our method for measuring amplification lies in comparing the intensity of the note obtained from a single receiving valve, plus rectifier, with the intensity obtained from two or more valves plus the same rectifier. The intensities are made equal by adjustment of the incoming energy, but this is not done by a variation of a mutual inductance, as in the method for audible frequencies, but by a variation in the intensity of the radio-frequency current at the source.

The arrangement is shown in Fig. 6, where *V* is the generating valve and L_1C_1 the anode oscillatory circuit. L_1 was of very low resistance. The grid inductance is L_3, but there was no tuned grid oscillatory circuit. A heater, *H*, with separate thermo-junctions, was included in the anode oscillatory circuit, and the current accurately measured. Several heaters were made and carefully adjusted in resistance, so that any one could be substituted for any other one in the oscillatory circuit without any measurable effect on the current or on the tuning. The same galvanometer was used throughout the experiments, but several heaters were necessary, as the current was varied from a few milliamperes to an ampere or more. For the heating wires used, the high frequency resistance was less than

0·5 per cent. greater than the steady current resistance, so that calibration of a combined heater and junction was easy. Variation in the generated current was brought about by altering the grid potential. In general, the first grid potential in a measurement corresponded to that of the point *A* and the second to that of a point *B*, which was adjustable. The change from *A* to *B* was brought about by a switch, and the position *B* was varied until equal intensity of received notes was obtained when one valve and rectifier was used as receiver for the *A* position and several valves and the rectifier for the *B* position. After obtaining the balance, the oscillatory currents in the anode circuit were read off at leisure. Their ratio was the value desired.

Previous experiments had indicated a small change of wave-length when the grid voltage was varied over a considerable range, and it was necessary to determine to what extent this was a disturbing influence. To do this, a secondary oscillatory circuit was set up and adjusted so that the heterodyne beat frequency was about 780 per second. After an adjustment of the grid voltage of the first valve circuit to reduce the oscillatory current to 1 per cent. of its former value, the heterodyne beat frequency was about 200 per second; but the note which was formerly the higher was now the lower, so that the change in radio frequency of the oscillations in the first circuit was about 980, corresponding to 0·1 per cent. However, in our experiments, the impedance in the receiving circuit L_4C_4 (Fig. 7) was appreciable, and both measurement and calculation showed the resonance curve to be such that the change in wave-length due to change of grid voltage of the transmitting circuit could have but a negligible effect on our measurements of amplifying power.

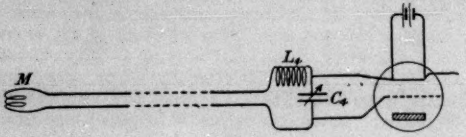


FIG. 7.

Direct radiation from the valve circuit to the amplifier set was in general appreciable, even when the distance between the sets was as great as 40 ft. Metal screens and compensating coils greatly reduced the action, and a few measurements were made on the assumption that direct radiation was negligible. However, in the method finally adopted, direct radiation was *not* excluded. The radio receiving circuit consisted of the coil M, the inductance L_4 and the condenser C_4 (Fig. 7). When this circuit was tuned and the oscillatory current in L_1C_1 was 1 ampere the noise in a receiving telephone, after amplification and rectification by three or four valves, could in general be heard very many yards away. It could be reduced by adjustment of the directions of the axes of L_1, L_3, M and L_4, and this was usually done. When the receiving circuit was thrown greatly out of tune the note was scarcely detectable, indicating that the receiving action of the valve leads, &c., was negligible. The axes of the coils L_1 and L_3 were made vertical, and that of L_4 horizontal. The coil M could be turned in any position.

The anode and grid characteristics of all the valves were found previous to any amplification measurements being made, and the grid potentials of the radio amplifying valves were adjusted for favourable amplification but poor rectification. The last valve in a set was of the same type as the others, but the grid potential of it was adjusted for favourable rectification.

Any method for measuring the amplification only of radio frequencies is, however, necessarily difficult, because in practically all cases when the filament currents and anode and grid potentials are adjusted for most favourable amplification there is also some rectification. Referring to Fig. 5, the part AB of the anode characteristic may be taken as a straight line, and the rate of change of the anode current with change of grid potential may be regarded as constant. If, therefore, the range of the oscillatory voltage lies within CD, and if these oscillatory voltages be symmetrical about the mid-point, there will be no rectification; but at C the grid current is zero, while at D it is 20 micro-amperes, and it rises with increasing rapidity as the grid potential is increased. If, therefore, the plate of the condenser C_4, which is connected to the grid, is charged positively by an amount corresponding to D, there will be a grid current, and therefore a leak; but when it is charged negatively by an amount corresponding to C there will be no grid current and no leak. Thus, it happens that in general the positive half of the incoming wave is cut down by a leak from the grid to a greater extent than the negative half. In other words, the potential variations impressed on the condenser and grid are not only cut down but are slightly rectified by the above action. In radio work, however, this is of little importance, and our measurements were made under practical conditions.

For comparison purposes, the standard note was that obtained from one amplifying valve and one rectifying valve, the telephone being included in the anode circuit of the latter or coupled to it by means of a suitable transformer. One or more radio-frequency valves and transformers were then added, and the resultant note reduced in intensity to that of the standard by reducing the current in the main oscillatory circuit. The ratio of the currents was taken as the amplification. The change from one condition to the other was made in a fraction of a second by means of a multiple brush switch, as in the method already described for audible frequencies. No change of tuning was made, and no interference of the main circuits was permissible, except for the adjustment of the grid potential of the valve *V*. The shunt resistance *R* was non-inductive. The values obtained with the radio-frequency valves were comparable with those for audible frequency. A few checks on the method were made, of which the following is an example : With two radio amplifiers and a current of 20 milliamperes in the circuit containing L_1C_1 a note of very moderate intensity was obtained. The current was increased five times to 100 milliamperes, and the note restored to its original intensity by shunting the telephone, of which the inductance and effective resistance were known at the audible frequency experimented with. The shunt value was such as to give a ratio of currents 4·8. This check was considered very satisfactory.

The writer desires to express his thanks to Mr. Albert Campbell and to Mr. D. W. Dye for many suggestions in connection with the measurements.

ABSTRACT.

The measurement of amplification at audible frequencies which is described is a mutual inductance method, the alternating voltage applied between the filament and grid of the first valve being varied by quickly changing the connections to an air-core transformer having two secondaries at right angles, which secondaries are capable of rotation so as to give any desired ratio of the mutual inductances between the primary and the respective secondaries. The source of alter-

nating voltage is a triode circuit having such capacity and inductance in the oscillatory portion as to give currents of a sonic frequency. The standard intensity of note is such as to enable signals to be read with comfort, and is obtained by adjustment of the source of alternating current. The first note is obtained when the telephone is in the anode circuit of an auxiliary valve, and the second when the telephone is transferred to the anode circuit of the last valve of the amplifying set, changes also being made in the applied alternating voltage, and in the connections between the valve circuits. All of these changes are brought about by a six-brush switch in a fraction of a second, so that readings are quickly obtained.

The method is compared with a modification of the shunted telephone method, and very good agreement is obtained when the amplifying values are comparatively low. For values of a high order such as 200, it is shown that the two methods are not likely to give the same results, the method described in the Paper being the better one.

In the measurement of amplification of a radio-frequency amplifier an undamped train of oscillations is broken up into groups of sonic frequency by shunting intermittently the condenser in the oscillatory circuit by an inductance of low resistance. A sonic train of oscillations is thus produced.

The receiving circuit is very similar to that used for audible frequency amplification, but the measurement of the amplifying power is made by quickly varying the current at the source, and not by a change of mutual inductance. between the two circuits.

DISCUSSION.

Prof. FORTESCUE said it was an advance to have a method involving the comparison of two sounds. Curiously enough, he had used this principle before the war with crystal detectors, and had given it up because of the difficulty of getting the two notes of similar character.

Mr. P. R. COURSEY (communicated): These two Papers by Messrs. Smith and Jordan bring forward some interesting points in connection with the measurement of the amplification factor of valves, and also raise the question as to the best method of carrying out such measurements. The arrangement used by Mr. Smith for the low-frequency measurement is particularly interesting, and should provide a very accurate method of carrying out such tests. The agreement obtained between the readings obtained and those given by the shunted telephone method also appears suprisingly good, although it is very doubtful whether the same accuracy is obtainable by the shunted telephone method as ordinarily employed in a wireless receiving station. The observed magnifications as given by the shunted telephone method are apparently less than those given by the first comparison method in the Paper for the larger amplifications. This is, of course, to be expected from the fact that when making measurements by the shunt method the amplitude of the oscillations is larger than in the first case, and therefore one would expect that the amplification would fall off owing to the bends of the characteristic curves. The shunted telephone may be looked upon rather as giving an indication of the " response " of any given detector or amplifier rather than a measure of its ultimate " sensitiveness."

As far as measurements of radio-frequency amplification are concerned, the method that is probably the most commonly used is that in which a steady source of oscillations is used at the transmitter and the coupling between the transmitter and receiver coils is varied in a known manner. With this method, as has been pointed out by Mr. Jordan, difficulties are experienced owing to the capacity between the two coils; but in general such difficulties can be overcome by a suitable earth screen, and, if necessary, by earthing the adjacent points of both transmitting and receiving coils. The screening of high-frequency C.W. oscillations is, however, by no means easy, as anyone who has worked with multi-valve amplifiers will readily appreciate.

Mr. Jordan's modification of the usual variable coupling method by providing a comparison for the sound intensity in the telephones in each case should lead to considerably greater accuracy as compared with the more usual arrangement; but it would seem possible that a somewhat simpler arrangement could be obtained by using two independent receiving search coils coupled to the transmitting coil (preferably on opposite sides of it), and arranging the induced oscillations in one of these to be detected by a single valve, while those from the other coil are passed through the amplifier. Both the receiving valve and the amplifier could be connected to the same telephones with a throw-over switch, and a rapid and accurate comparison could then be made between the intensities of the resulting sounds in the telephone in each case, thus giving the direct measure of the amplification obtained by adjusting the couplings to the two search coils until equality of signal strengths is obtained.

An Instrument for Indicating the Hardening Temperature of Steel. By the Hon. C. W. STOPFORD *and* C. R. DARLING.

(EXHIBITED AT THE MEETING ON JANUARY 23, 1920.)

THE instrument depends upon the fact that the best results are secured by quenching steel at a temperature just above that at which it demagnetises, and is used in conjunction with an electric resistance furnace with inductive winding. A pivoted soft-iron rod is surrounded by a coil joined in series with the furnace winding. The deflection of the iron rod in either direction moves an upright spring, and breaks the circuit of a bell or buzzer. In using, the instrument is placed near the furnace, and when the current is switched on the pointer deflects and breaks the bell circuit. By a screw movement the springs are restored to such a position that the bell just rings. On placing the steel articles in the furnace the increased flux causes the bell circuit to be broken; but on demagnetising, the pivoted iron rod comes back to its original position, causing the bell to ring. The indicator may be used with either direct or alternating current, and responds when only a small charge of steel is placed in the furnace.

A Thermo-electric Cell of Constant Voltage. By C. R. DARLING.

(EXHIBITED AT THE MEETING ON JANUARY 23, 1920.)

THIS cell depends upon the constancy of the E.M.F. developed by molten bismuth and aluminium over a wide range of temperature (Darling and Grace, "Proceedings," Vol. XXIX. Part I.). An iron vessel is provided with a side branch of silica tubing, about 10 in. long, both being filled with bismuth. An aluminium wire covered with graphite passes through the lid of the vessel into the molten bismuth, and forms the hot junction, the cold junction being similarly formed at the remote end of the silica tube, and kept in oil in a vacuum flask. On placing a burner beneath the vessel the E.M.F. rises to 15 micro-volts at the melting point of bismuth (267°C.), the cold junction being at 15°, at which voltage the cell remains constant over a further 300°. It has been found useful as a closely approximate standard in measuring thermal E.M.F.s, and as a "set-up" against another E.M.F. when using a pyrometer indicator for measuring special ranges. The internal resistance of the instrument exhibited is approximately 0·5 ohm.

XV. *The Influence of Small Changes of Temperature on Atmospheric Refraction.* *By* SIR ARTHUR SCHUSTER, *F.R.S.*

RECEIVED FEBRUARY 4, 1920.

1. It has been suggested that the changes of temperature caused by the moon's shadow sweeping through our atmosphere may have an appreciable effect on optical refraction, and hence on the apparent position of stars near the sun. I propose to examine the nature of the effect and its order of magnitude. The investigations will be sufficiently general to apply also to other cases where the air is affected by systematic changes of temperature.

The optical length of a ray of light passing through a gas depends mainly on the total mass of gas traversed. If μ denote the refractive index, and ρ the density $(\mu-1)/\rho$ remains sensibly constant, so that a small change $\delta\rho$ increases the refractive index by $(\mu-1)\delta\rho/\rho$. Inequalities of pressure at two adjacent points of a horizontal surface are quickly removed by convection currents. The differences between the refractive indices at two such points will, therefore, be mainly determined by the differences in temperature, and a small fall in temperature τ will be connected with a change $\delta\rho$ in density by the relation $\delta\rho/\rho=\tau/t$, the temperature being measured on the absolute scale. Along each horizontal surface we have

$$\delta\mu=(\mu-1)\tau/t=(\mu_0-1)\rho\tau/\rho_0 t, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where μ_0, ρ_0 are the values of μ, ρ at some standard temperature—*e.g.*, 0°C. For the change of density with altitude I shall adopt the exponential formula, so that at altitude y the density is $\rho e^{-h_1 y}$.

The variation of temperature along the surface of the ground in the direction of the path of the moon's shadow is represented with sufficient accuracy by

$$\tau=\tfrac{1}{2}\tau_0(1+\cos m(x+a)), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

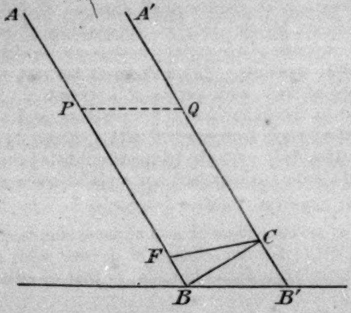
when x is measured from the centre of the umbra, and π/m is the radius of the penumbra, τ_0 denoting the maximum *fall* of temperature. The length a is introduced to allow for the displacement of the point of lowest temperature, which generally occurs at, or even after, the end of totality. The complete expression for τ would require additional terms involving multiples of mx; they might easily be taken into account if high accuracy were aimed at.

There is greater difficulty in estimating the influence of altitude on the changes of temperature. The calculation is simplified by introducing the factor te^{-h_2y}/t_0 for the purpose, and by a proper choice of h_2 we may obtain an approximation to the variation which in any case will be much affected by local circumstance. We must finally provide for a possible change of phase, which may be taken to be proportional to the altitude. Irrelevant complications are avoided by treating the problem as two-dimensional—*i.e.*, by assuming the central line of the moon's shadow to pass through the centre of the earth.

Writing $h=h_1+h_2$, we then obtain

$$\rho\tau/\rho_0 t=\tfrac{1}{2}\tau_0 e^{-hy}(1+\cos .\, m(x+a-y\tan\theta)), \quad . \quad . \quad . \quad . \quad (3)$$

where θ is the angle between the vertical and the line joining the minima of temperature at different levels. This line may for convenience be called the line of slope of temperature.



Combining this with (1), we find

$$t_0\delta\mu=\tfrac{1}{2}(\mu_0-1)\tau_0 e^{-hy}(1+\cos .\, m(x+a-y\tan\theta)). \quad . \quad . \quad (4)$$

Let AB be a ray of light which will be slightly curved owing to the normal atmospheric refraction, and let $A'B'$ be an adjacent ray coming from the same distant source of light. The points A and A' are taken on the same horizontal lines at a sufficient elevation to allow us to disregard any change of refractive index. The wave-front BC is at right angles to AB. If the refractive index along PQ be raised from μ to $\mu+\delta\mu$ the excess of $\delta\mu$ at Q above that at P is $s\, d\delta\mu/dx$, s

being the distance PQ. The total increase of optical length of $A'B'$ over that of AB is

$$f=\int_0^\infty \frac{d\,.\,\delta\mu}{dx}\,dl,$$

where l is measured along AB. To correct for this increase, the wave-front is pushed back through a distance $BF=\mu_0 f$, where we may take the value of μ_0 to be essentially equal to unity. The angle ε through which the ray is turned is now determined by

$$-\tan\varepsilon=CF/BC=s^{-1}\sec\gamma\int_0^\infty \frac{d\delta\mu}{dx}\,dl,$$

where the negative sign indicates a diminution of the angle between the ray and the vertical; or, substituting (4)

$$\tan\varepsilon=Cm\tau_0\sec\gamma\int_0^\infty e^{-hy}\sin m(x+a-y\tan\theta). \quad . \quad . \quad (5)$$

C is a numerical factor equal to $(\mu_0-1)/2\mu_0\tau_0$, which for atmospheric air is $5{\cdot}4\times10^{-7}$. In deducing the deflection of a ray produced by small changes in the refractive index at different points, we have measured the changes of optical length along the original path of the ray; this is allowable so long as the altered path lies close to it. There is, therefore, here a tacit assumption that the deviation is small, and we may substitute ε for $\tan\varepsilon$, with the reservation, however, that the result will only be correct so long as ε^2 is negligible.

Before performing the integration we substitute

$$l=y\sec\gamma,\ x=x_0+y\tan\gamma.$$

Here x_0 denotes the distance of the point of observation from the centre of the umbra. Strictly speaking, the upper limit of the integral should not be infinite, because (2) holds only as far as $x=\pm\pi/m$, τ being zero for larger values of x; but considering the numerical value of h and m the error committed is negligible.

The expression for the integral is simplified by the following substitutions:—

$$c=x_0+a\,;\ \varphi=\gamma-\theta\,;\ n=m\sec\theta\sec\gamma\,;\ \tan\psi=n\sin\varphi/h.$$

We then obtain from (5)

$$\varepsilon=Cm\tau_0\sec^2\gamma\sin(mc+\psi)/(h^2+n^2\sin^2\varphi)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (6)$$

This equation contains the solution of our problem. If we disregard the diminution of density with altitude, which is equivalent to assuming zero value for h, we obtain Prof. Anderson's result (" Nature," December 4, 1919).* In that case ψ is a right angle, and at the point of minimum temperature ($c=o$) the deflection ε is inversely proportional to $\sin \varphi$. The angle φ is that formed by the line of sight with the line marking the maximium fall of temperature at different altitudes. The latter is assumed by Prof. Anderson to be the central line of the moon's shadow—*i.e.*, the line drawn to the centre of the sun—so that φ is the angular distance of a point near the sun's limb from its centre. If Prof. Anderson is right, the displacement of a star near the sun during an eclipse would, therefore, be inversely proportional to its distance from the sun's centre in accordance with the result of observation. But even with an unlimited atmosphere all would depend on a very delicate adjustment of temperature distribution. If—as would be more natural to assume—the fall of temperature were regulated not by the central line of the moon's shadow, but by the edge of the umbra, the result would be entirely different ; ε would cease to be a small angle, and our equations would not be applicable.

But these considerations are beside the point, for in the denominator of (6) it is the second and not the first term that is dominating. The exponential factor of h is made up of two parts, of which h_1 depends on the diminution of density and h_2 on the diminution of the cooling effect with altitude. Neglect for a moment the latter—*i.e.*, assume the fall of temperature to be the same at all levels. In an isothermic atmosphere h would have the value of 0·125 with the kilometre as the unit of length. Taking the diameter of the penumbra to be 300 km. m is approximately 0·02, and hence mh_1^{-1} is 0·16. If the diminution of the temperature effect be taken into account, I estimate that this figure should be divided by four, and unless the sun be very near the horizon mh^{-1} will be considerably less than unity. We may therefore disregard $n^2 \sin^2\varphi$ in the denominator, as compared with h^2.

What interests us is not the actual amount of refraction, but its differences at different points near the sun, and for this purpose the best condition for a favourable effect is that made by Prof. Anderson with regard to the slope of temperature in the atmosphere. We continue to assume, therefore, that θ is the zenith distance of the centre of the sun—in other words, that φ is the angular distance from the sun's centre.

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* Since the above was written Prof. Anderson has modified his position ("Nature," Jan. 29, 1920) and some of the criticisms applicable to his original treatment are no longer valid.

The angle ψ being small, we may substitute the sine for its tangent, and calling ε' that part of ε which depends on φ we find

$$\varepsilon' = Cmnh^{-2}\tau_0 \cos mc \sin \varphi. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When the limited height of the atmosphere is taken into account, the displacement of a star therefore increases with increasing distance from the sun, in contradiction to the observed effects. For the purpose of estimating the total amount of the effect, we may take the sun to be at an altitude of 45 deg., and the maximum fall of temperature to be 5°C. At or near the point of maximum cooling ($c=o$), (7) then becomes

$$\varepsilon' = 20Cm^2 \sin \varphi / h^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$= (2{\cdot}58 \times 10^{-8}) \sin \varphi\,;$$

or, if φ is small,

$$\varepsilon'/\varphi = 2{\cdot}58 \times 10^{-8}.$$

If the distance of a star from the centre of the sun be a solar diameter—*i.e.*, 0·5 deg.—the displacement in seconds of arc is $4{\cdot}6 \times 10^{-6}$. The distance of such a star from another one equally far away on the other side of the sun would be measured by about the 230,000th part of a second.

Apart from the deflexion depending on the angular distance, there is a portion depending only on the zenith distance of the sun. This is, of course, in addition to the ordinary refraction when there is no eclipse. According to (6) this portion is greatest when sin mc is unity—*i.e.*, when the observer stands at the point where the depression of temperature alters most quickly. As ψ is a small angle, the deflexion then is for an altitude of 45 deg. : $2Cm\tau_0/h$, which amounts to about the 50th part of a second. Owing to differences in γ, the distances of two stars, one of which is above and the other below the sun, would have their apparent distances increased by an amount appreciably greater than that depending on φ.

I have also carried out the calculation for the case where the diminution of the temperature effect with altitude is represented by a linear function of h and proportional to $h-r$, vanishing at a height r. The expressions obtained are more complicated, but show that the adopted value for the diminution of the temperature effect is equivalent to the assumption that if diminishing linearly it vanishes at an altitude of 6 km.

ABSTRACT.

The Paper is an investigation of the possible deviation of the light from a star near the sun due to the temperature changes in the atmosphere produced by the passage of the moon's shadow across the earth during an eclipse. It is shown that while the actual displacements from this cause vary widely for slight differences in the assumed conditions, they are always negligibly small compared with the effects observed at the last solar eclipse.

DISCUSSION.

Mr. T. SMITH said he had not grasped the necessity of assuming that the phenomena would be symmetrical about the centre of the sun. If this limitation were dispensed with in the treatment, much larger displacements might be obtained.

The AUTHOR said the symmetry was necessary to account for the actual observations.

Dr. CHREE said he was not convinced that the actual observations justified the assumption of symmetry. Of the 15 or so stars on the plate, probably only five or six were of significance, and he did not think that they gave unquestionable evidence of symmetry in the displacements. Moreover, during any single eclipse, the time of day is changing, and it is impossible to get the unmixed temperature effects of the eclipse alone. A large number of eclipse observations would have to be made and compared before the evidence was regarded as conclusive. Further, he thought such observations should be made from a considerable height, as observations taken near the surface of the earth were always subject to variable local disturbances.

Mr. F. J. W. WHIPPLE said that he did not see the justification for assuming that the pressure could be regarded as uniform. Convection could hardly equalise the pressure in the time available. It might, in fact, increase the difference. The shadow traversed the earth at a very high speed, and a pressure wave might be sent out. Temperature effects might also arise from the passage of this adiabatic disturbance through the atmosphere.

The AUTHOR agreed that the assumption of equality of pressure presented a difficulty; but he thought convection would undoubtedly reduce any differences rather than increase them.

XVI. *On Balancing Errors of Different Orders.* *By* T. SMITH, *B.A.* (*From the National Physical Laboratory.*)

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THE problem to be considered may be illustrated by a somewhat artificial example. Suppose it were intended to calculate $\cos\theta$ for values of θ not exceeding one radian from an expression containing no power of θ higher than the sixth. The use of the leading terms of the expansion in infinite series

$$1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}$$

or

$$1-0\cdot 5\theta^2+0\cdot 041666\ldots\theta^4-0\cdot 0013888\ldots\theta^6$$

involves errors amounting to about 250 seventh place units. On the other hand, the use of the expression

$$0\cdot 999{,}999{,}8-0\cdot 499{,}993{,}7\theta^2+0\cdot 041{,}635{,}2\theta^4-0\cdot 001{,}339{,}9\theta^6$$

only involves errors of about 2 in the seventh place. The alterations here introduced in the numerical coefficients of the first series have the effect of compensating with fair precision for the omission of the terms in θ^8 and θ^{10}, the only other terms of importance when seven significant figures are retained. It is important for many purposes to know how the effect of such higher order terms may be most closely imitated by a series of terms of lower orders, and what magnitude the outstanding errors may reach. The series to be considered may be assumed to consist of terms involving only positive integral powers of the independent variable.

A particular case of the more general problem may be considered in the first place. Let it be required to find the unknown coefficients a_1, a_2, a_3 . . . a_{n-1} in terms of the known quantities a_n, x_1, x_2, so that

$$y=a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n \quad \ldots \quad (1)$$

lies between the closest possible limits as x varies continuously from x_1 to x_2. By substituting z for x, where z is defined by the equation

$$x-x_1=z(x_2-x_1),$$

the limits for the independent variable become 0 and 1 without any alteration in the form of y. These limits will be assumed in the following discussion.

It is not unnatural to suppose that as x changes from 0 to 1 the variation in y will be least when y is made to oscillate repeatedly between two fixed values whose difference determines the required range. The case in which all the stationary points on the curve lie between $x=0$ and $x=1$ may be considered. The limiting values for y (or one of them) are necessarily reached at the extreme points $x=0$ and $x=1$ when the range is a minimum. For, if not, y only passes through the limiting values when $x=-\varepsilon$ and $x=1+\eta$, where ε and η are positive. The limits must be reached for finite values of x, since y tends to infinity with x. Under the conditions specified a new relation can be found of the same type as (1), say $\eta=a_n\xi^n+$ a series of lower powers of ξ, where

$$\xi(1+\varepsilon+\eta)=x+\varepsilon$$

and

$$\eta(1+\varepsilon+\eta)^n=y,$$

so that η lies between smaller limits than y as ξ varies from 0 to 1 corresponding to a variation of x between $-\varepsilon$ and $1+\eta$. It follows that for minimum variation in y a limiting value is reached at both $x=0$ and $x=1$.

The two limits for y, which may be denoted by a and $-a$, since the constant term a_0 determines only the centre of the range, and does not influence its width, will be reached alternately as x increases. The values of x at which they are attained may accordingly be denoted by

$$0,\ x_2,\ x_4,\ x_6\ \ldots$$

in the one case, and by

$$x_1,\ x_3,\ x_5,\ x_7\ \ldots$$

in the other.

Assuming that the values $y=\pm a$ are attained the greatest possible number of times, y may be defined by the equations

$$y-a=a_nx(x-x_2)^2(x-x_4)^2(x-x_6)^2\ldots(x-x_{n-2})^2(x-1) \quad (2)$$

$$y+a=a_n(x-x_1)^2(x-x_3)^2(x-x_5)^2\ldots(x-x_{n-1})^2 \quad (3)$$

if n is even, or by

$$y-a=a_nx(x-x_2)^2(x-x_4)^2(x-x_6)^2\ldots(x-x_{n-1})^2 \quad (4)$$

$$y+a=a_n(x-x_1)^2(x-x_3)^2(x-x_5)^2\ldots(x-x_{n-2})^2(x-1) \quad (5)$$

if n is odd. The condition that the two equations of each pair determine the same function suffices to find x_1, x_2, $x_3\ldots$ and consequently the function itself, uniquely. For in each pair of equations there are n unknowns a, x_1, x_2, $\ldots x_{n-1}$, and the curve given by these equations passes through $2n$ points. The conditions, therefore, just suffice to determine all the unknowns. The equations for y can easily be found from this consideration in any particular case, but for the general discussion of the problem it is preferable to derive a solution by forming the differential equation satisfied by y. Since y has in $x_1, x_2, x_3, \ldots x_{n-1}$ the full number of stationary points on the curve

$$\frac{dy}{dx}=na_n(x-x_1)(x-x_2)\ldots(x-x_{n-2})(x-x_{n-1}) \quad . \quad (6)$$

Thus, whether n be even or odd, from equations (2) to (6) it is seen that the relation

$$n^2(a^2-y^2)=x(1-x)\left(\frac{dy}{dx}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

is always satisfied. Differentiate this equation once and remove the common factor $\frac{dy}{dx}$. The resulting linear differential equation is

$$x(1-x)\frac{d^2y}{dx^2}+(\tfrac{1}{2}-x)\frac{dy}{dx}+n^2y=0. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is a particular case of the equation satisfied by hypergeometric series. Denoting as usual the series

$$1+\frac{a \cdot b}{1 \cdot c}x+\frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3c(c+1)(c+2)}x^3+\ldots$$

by $F(a, b, c, x)$, and remembering that the solution with a finite number of terms is alone admissible, the value found for y is

$$y=cF(n, -n, \tfrac{1}{2}, x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$=c\left\{1-\frac{n^2}{2!}4x+\frac{n^2(n^2-1^2)}{4!}(4x)^2-\frac{n^2(n^2-1^2)(n^2-2^2)}{6!}(4x)^3+\ldots\right\} \quad (10)$$

where c is a constant. Written as a descending series the value is

$$y=a_nx^nF(\tfrac{1}{2}-n, -n, 1-2n, x^{-1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$=a_nx^n\left\{1-2n(4x)^{-1}+\frac{2n(2n-3)}{2!}(4x)^{-2}\right.$$
$$-\frac{2n(2n-4)(2n-5)}{3!}(4x)^{-3}+\frac{2n(2n-5)(2n-6)(2n-7)}{4!}(4x)^{-4}$$
$$\left.-\frac{2n(2n-6)(2n-7)(2n-8)(2n-9)}{5!}(4x)^{-5}+\ldots\right\}. \quad . \quad (12)$$

From a comparison of the coefficients in (10) and (12), or from the values of (9) and (11) when $x=1$, it is seen that the coefficients c and a_n are connected by the relation

$$(-4)^n c = 2a_n. \qquad (13)$$

Equation (8) is unaltered if $1-x$ be written for x, and (9) can therefore be expressed as

$$y = c'F(n, -n, \tfrac{1}{2}, 1-x) \qquad (14)$$

The two forms (9) and (14) must be simple transformations of one another. Since the differential coefficient of

$$F(a, b, c, x) \text{ is } \frac{ab}{c} F(a+1, b+1, c+1, x),$$

it follows from the comparison of (9) and (14), and from the fact that all the coefficients are alternately positive and negative, none being zero, that all the stationary points of y are real, and lie between $x=0$ and $x=1$. The function y therefore satisfies all the conditions that have been laid down. The range of variation is $2a$, where $a=c$, the value for $x=0$. Thus, from (13) the range is that for the term in x^n alone multiplied by 4, or, apart from sign, 4^{1-n}.

It has now to be shown that y as just found lies between closer limits than any other function of the form (1) having the same coefficient for the term in x^n. The curve obtained has n branches between the limits $y=\pm a$, $x=0$ and $x=1$, the terminal points of the branches being the $n-1$ stationary points and the end points. If Y is another function varying if possible between closer limits $\pm\beta$ from $x=0$ to $x=1$, this curve must intersect every branch of the y curve. Thus, $Y-y=0$ for n values of x; but $Y-y$ is at most of degree $n-1$ in x, and thus can only vanish for n values of x if every term is zero. The limits $\pm a$ must, therefore, be exceeded at some part of the interval $x=0$ to $x=1$ by any other function of the form prescribed.

In a number of cases the problem is modified by the consideration that only odd or only even powers of the variable are admissible. The latter case is evidently solved at once by writing x^2 for x and $n/2$ for n. The discussion may proceed generally on the lines already followed, except that it may be noted that the restriction imposed means that the limits for $x=0$ to 1 will also hold for $x=0$ to $x=-1$, with or without a change of sign for corresponding distances on opposite sides of $x=0$, according as the powers are odd or even. The case may evidently be considered by writing $x=1$ and $x=-1$ as the limits at which the values $y=\pm a$ are to be attained. The differential equation is therefore derived from

$$n^2(a^2-y^2)=(1-x^2)\left(\frac{dy}{dx}\right)^2,$$

giving

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

which may be re-written in the form

$$\xi(1-\xi)\frac{d^2y}{d\xi^2}+(\tfrac{1}{2}-\xi)\frac{dy}{d\xi}+\frac{n^2}{4}y=0,$$

where ξ stands for x^2. When n is even this gives

$$y=cF\left(\frac{n}{2},\ -\frac{n}{2},\ \tfrac{1}{2},\ x^2\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

agreeing with the solution already suggested. This solution does not satisfy the conditions of the problem when n is odd, the number of terms being infinite. The solution to be adopted in this case is

$$y=cxF\left(\frac{1+n}{2},\ \frac{1-n}{2},\ \frac{3}{2},\ x^2\right), \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

which has the required number of terms and is easily seen to satisfy equation (15).

It should be particularly noted that an expression like (17), though giving the minimum variation over the interval -1 to 1, does not give the least range for a series of odd powers over the interval 0 to 1. For take the simple case

$$y=3x^5-5x^3(a^2+b^2)+15xa^2b^2 \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

which is stationary at the points

$$(a,\ \alpha),\ (b-\beta),\ (-a,\ -\alpha),\ (-b,\ \beta)$$

where $b>a$ and a, b, α are positive, and

$$\alpha=2a^3(5b^2-a^2),\ \beta=-2b^3(b^2-5a^2). \quad . \quad . \quad . \qquad (19)$$

Consider in conjunction with (18) the associated function

$$\eta=3\xi^5-5\xi^3(a^2+b^2)\lambda^2+\frac{15}{4}\xi(b^2-a^2)^2\lambda^4 \quad . \quad . \quad . \qquad (20)$$

which has the stationary points

$$\left\{(b-a)\frac{\lambda}{\sqrt{2}},\ (\alpha+\beta)\frac{\lambda^5}{\sqrt{2}}\right\},\quad \left\{(b+a)\frac{\lambda}{\sqrt{2}},\ -(\alpha-\beta)\frac{\lambda^5}{\sqrt{2}}\right\}$$

$$\left\{-(b-a)\frac{\lambda}{\sqrt{2}},\ -(\alpha+\beta)\frac{\lambda^5}{\sqrt{2}}\right\},\quad \left\{-(b+a)\frac{\lambda}{\sqrt{2}},\ (\alpha-\beta)\frac{\lambda^5}{\sqrt{2}}\right\}.$$

The conditions for the end points are evidently

$$\alpha=3-5(a^2+b^2)+15a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and

$$(\alpha+\beta)\frac{\lambda^5}{\sqrt{2}}=3-5(a^2+b^2)\lambda^2+\frac{15}{4}(b^2-a^2)^2\lambda^4. \quad . \quad . \quad (22)$$

These equations show that if one solution of the kind considered is known another may be derived directly. The solution of the type hitherto considered is

$$a=\frac{\sqrt{5}-1}{4},\ b=\frac{\sqrt{5}+1}{4},\ \alpha=\beta=\frac{3}{16},$$

so that y covers the range 0·375 as x increases from 0 to 1. A comparison of the solutions of (18) and (20) shows that a self-conjugate solution is obtained when $\lambda=1$, so that

$$b=a(\sqrt{2}+1),\ \alpha=\beta(\sqrt{2}+1),$$

the numerical values being $a=0{\cdot}33741$, $b=0{\cdot}81459$, $\alpha=0{\cdot}24615$, $\beta=0{\cdot}10196$, and the total range of variation 0·34811, which is less than in the previous solution. The principles followed in the foregoing discussion make it clear that the variation over the half range will be a minimum when one of the limits is zero, the value for $x=0$. The solution of (20) shows that this solution may be derived from the first solution of (18), since $\alpha=\beta$. The value of λ from (22) is 1·038116, giving stationary points at (0·36703, 0·31970) and (0·82070 ,0), with a range of variation 0·31970.

The expressions which have been found having the property of oscillating about zero with small excursions may be used to simplify expressions for purposes of calculation, or may be applied in a variety of ways in finding the most favourable combination of errors of different orders which may be deliberately introduced to compensate as far as possible for the presence of an irremovable error. The results already reached show that by this means the outstanding errors are greatly reduced, in many cases to an extent which renders them immeasurable. In such applications a slight variation from the foregoing cases is sometimes necessary, as the conditions of the problem involve that, though both odd and even powers of the variable are admissible in general, the first power must not be present. This requirement is met by transferring the origin from $x=0$ to $x=x_1$, the first stationary point, and replacing x as independent variable by ξ, where

$$\xi=(x-x_1)/(1-x_1).$$

It is of interest in this connection to know where the stationary values occur. They are given in the following table from $n=2$ to $n=10$.

Values of xf or which y is stationary.

n									
2.	·5								
3.	·25	·75							
4.	·14645	·5	·85355						
5.	·09549	·34549	·65451	·90451					
6.	·06699	·25	·5	·75	·93301				
7.	·04952	·18826	·38874	·61126	·81174	·95048			
8.	·03806	·14645	·30866	·5	·69134	·85355	·96194		
9.	·03015	·11698	·25	·41318	·58682	·75	·88302	·96985	
10.	·02447	·09549	·20611	·34549	·5	·65451	·79389	·90451	·97553

It will be noted from the table that the stationary points are closely packed towards the ends of the range, and are widely separated near the middle. The curves thus differ very appreciably from those representing the circular functions when plotted against the angle. The calculation of the roots is simplified by noticing that if n has integral factors the equation may be resolved into a series of equations of lower orders. Thus, if $n=pq$, the qth, $2q$th, $3q$th . . . stationary points occur for the same values of x as the 1st, 2nd, 3rd . . . points in the p equation. That these results are generally true may be seen as follows :—

Let y denote $F(p, -p, \frac{1}{2}, x)$, which oscillates between $\pm a$, so that

$$p^2(a^2-y^2)=x(1-x)\left(\frac{dy}{dx}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and

$$x(1-x)\frac{d^2y}{dx^2}+(\tfrac{1}{2}-x)\frac{dy}{dx}+p^2y=0. \quad . \quad . \quad . \quad . \quad (24)$$

Further, let z denote $F(q, -q, \frac{1}{2}, y)$, transformed for the interval $y=-a$ to $y=+a$, so that by analogy with (15) z satisfies the equation

$$(a^2-y^2)\frac{d^2z}{dy^2}-y\frac{dz}{dy}+q^2z=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The variable z may be expressed as a function of x instead of y by eliminating a and y from the above three equations together with

$$\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$$

and

$$\frac{d^2z}{dx^2}=\frac{d^2z}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2+\frac{dz}{dy}\cdot\frac{d^2y}{dx^2}.$$

The elimination of a between (23) and (25) yields

$$x(1-x)\frac{d^2z}{dy^2}\left(\frac{dy}{dx}\right)^2-p^2y\frac{dz}{dy}+p^2q^2z=0,$$

or

$$x(1-x)\frac{d^2z}{dx^2}+(\tfrac{1}{2}-x)\frac{dz}{dx}+p^2q^2z$$

$$=\frac{dz}{dy}\left\{x(1-x)\frac{d^2y}{dx^2}+(\tfrac{1}{2}-x)\frac{dy}{dx}+p^2y\right\}=0\quad . \qquad (26)$$

by (24), so that z is the polynomial corresponding to $n=pq$. The same result follows more simply by combining the two equations corresponding to (7). Now, z may be regarded as a function of y only, and is thus stationary when y is stationary. Therefore values of x for which y is stationary have the same property for z. It is easily seen that the determination of all the stationary positions for $F(n, -n, \frac{1}{2}, x)$, where n is not prime, may be resolved into the solution of a series of equations of the order of the lowest integral factor of n other than unity. In the case just considered there are q values of x on each branch of the p curve for which z is stationary. The roots may also be found simply by solving a trigonometrical equation.

The formulæ that have been obtained may be applied repeatedly to remove any number of terms from an expansion, and thus give an approximate simplified expression for a given function. As, however, the roots of $F(n, -n, \frac{1}{2}, x)$ depend upon n, the expression so obtained will not in general be the closest possible approximation to the function of that particular form; but if the series from which the approximation is derived converges rapidly the stationary positions will not differ greatly from those of the polynomial used to remove the lowest term. The knowledge of the roots of the corresponding hypergeometric series may be utilised in obtaining the closest approximation.

Assume that it is required to determine the coefficients $a_0, a_1, a_2 \dots a_n$, so that the polynomial

$$a_0+a_1x+a_2x^2+ \dots +a_nx^n$$

may represent the function $\varphi(x)$ as closely as possible throughout the interval $x=\alpha$ to $x=\beta$. It may be assumed that $\varphi(x)$ has no stationary values within this interval. The argument already applied shows that y, defined by the equation

$$y=\varphi(x)-a_0-a_1x-a_2x^2 \dots -a_nx^n$$

must be made to lie between the values $\pm\gamma$, that one or both of these values must be reached for $x=\alpha$ and $x=\beta$, and that between these limits there must be n values of x, say $x_1, x_2, x_3 \dots x_n$, at which $y^2=\gamma^2$ and $\frac{dy}{dx}=0$. Thus, writing φ for $\varphi(x)$, φ_1 for $\varphi(x_1) \dots$ and denoting differentiations by accents, the $2n+2$ conditions

$$\begin{aligned}
\varphi(\alpha)-\gamma &= a_0+a_1\alpha+a_2\alpha^2+ \dots +a_n\alpha^n \\
\varphi_1+\gamma &= a_0+a_1x_1+a_2x_1^2+ \dots +a_nx_1^n \\
\varphi_1' &= a_1+2a_2x_1+ \dots\dots +na_nx_1^{n-1} \\
\varphi_2-\gamma &= a_0+a_1x_2+a_2x_2^2+ \dots +a_nx_2^n \\
\varphi_2' &= a_1+2a_2x_2+ \dots\dots +na_nx_2^{n-1} \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\varphi(\beta)\pm\gamma &= a_0+a_1\beta+a_2\beta^2+ \dots +a_n\beta^n
\end{aligned}$$

suffice to determine $a_0, a_1 \dots a_n, x_1, x_2 \dots x_n$, and γ when φ, α and β are given. The polynomial formed by substituting these values for $a_0, a_1 \dots a_n$ is the closest possible approximation to φ of the assigned form for the interval α to β.

The method as it stands is in most cases unsuited to numerical computation. A rigorous method of treatment is possible if over the whole interval the errors resulting from the application to the usual expansion of the approximate method already described have been found. There will ordinarily be $n+2$ positions where these attain considerable dimensions, — the two end points and n intermediate stationary values. Let the magnitudes of these errors and their positions be noted. If these errors are all of equal numerical value, but alternating in sign, the expansion obtained is the closest possible fit. Usually they vary in magnitude. It is, however, possible to write down at once a correcting expression which will make the values at these points of the required magnitude while retaining the proper form for the approximate expression. When this is added the stationary points for the sum will be slightly displaced ; but as long as the displacements are small the values at the points previously considered will not differ appreciably from those at the new stationary points, and the problem is solved to a sufficient degree of accuracy. If the errors found at $\alpha, x_1, x_2 \ldots$ be denoted by $\varepsilon_0, \varepsilon_1, \varepsilon_2 \ldots$, the correcting factor η is readily seen to be given by the equation

$$\begin{vmatrix} -\eta & 0 & 1 & x & x^2 & \ldots & x^n \\ \varepsilon_0 & 1 & 1 & \alpha & \alpha^2 & \ldots & \alpha^n \\ \varepsilon_1 & -1 & 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \varepsilon_2 & 1 & 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \varepsilon_3 & -1 & 1 & x_3 & x_3^2 & \ldots & x_3^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

In most cases a simplification is possible, as the positions of $x_1, x_2 \ldots$ are very close to the stationary positions of $F(n+1, -n-1, \frac{1}{2}, x)$. If $-\eta_1$ denotes

$$\varepsilon_0 \frac{(x-x_1)(x-x_2) \ldots (x-x_n)(x-\beta)}{(\alpha-x_1)(\alpha-x_2) \ldots (\alpha-x_n)(\alpha-\beta)}$$
$$+\varepsilon_1 \frac{(x-\alpha)(x-x_2) \ldots (x-x_n)(x-\beta)}{(x_1-\alpha)(x_1-x_2) \ldots (x_1-x_n)(x_1-\beta)},$$
$$+ \ldots$$

it is evident that η_1 assumes the values $-\varepsilon_0, -\varepsilon_1, -\varepsilon_2 \ldots$ at $\alpha, x_1, x_2 \ldots$ and would thus remove the existing errors at these points. This expression cannot, however, be used as a correcting factor, since it is of the order $n+1$ in x. If $x_1, x_2, x_3 \ldots x_n$ may be assumed to be the stationary points of $F(n+1, -n-1, \frac{1}{2}, x)$, η may be at once reduced to the next order without violating the other conditions by adding a suitable multiple of this function. The correcting term is thus

$$-\Sigma\varepsilon_\lambda \frac{(x-\alpha)(x-x_1) \ldots (x-x_{\lambda-1})(x-x_{\lambda+1}) \ldots (x-x_n)(x-\beta)}{(x_\lambda-\alpha)(x_\lambda-x_1) \ldots (x_\lambda-x_{\lambda-1})(x_\lambda-x_{\lambda+1}) \ldots (x_\lambda-x_n)(x_\lambda-\beta)}$$
$$-\frac{1}{2(-4)^{-n}}(F)\,\Sigma \frac{\varepsilon_\lambda}{(x_\lambda-\alpha) \ldots (x_\lambda-x_{\lambda-1})(x_\lambda-x_{\lambda+1}) \ldots (x_\lambda-\beta)}$$

if (F) denotes $F\left(n+1, -n-1, \frac{1}{2}, \frac{x-\alpha}{\beta-\alpha}\right)$

where α and β are usually 0 and 1 respectively.

From the high order of accuracy attained in typical examples, it is a simple deduction that the mathematical expression of a law discovered from experimental results, in spite of a high order of agreement, cannot safely be taken as a true representation of the facts unless the agreement extends over a very great interval.

It is evident that the expansion of an arbitrary function $\varphi(x)$ in the form

$$\varphi(x)=a_0F_0+a_1F_1+a_2F_2+\ .\ .\ .$$

where F_n denotes $F(n,\ -n,\ \frac{1}{2},\ x)$ may prove of considerable importance. To derive such an expansion the identity

$$\tfrac{1}{2}(4x)^n=\frac{(2n)!}{(n!)^2}F_0-\frac{(2n)!}{(n-1)!(n+1)}F_1$$
$$+(-)^p\frac{(2n)!}{(n-p)!(n+p)!}F_p+\ldots+(-)^nF_n. \quad (27)$$

may be employed. The special convention $F_0=\frac{1}{2}$ has been introduced. If then $\varphi(x)$ is required from $x=h$ to $x=h+1$, Taylor's theorem gives

$$\varphi(h+x)=\varphi+x\varphi'+\frac{x^2}{2!}\varphi''+\frac{x^3}{3!}\varphi'''+\ .\ .\ .$$

where the general functions on the right depend upon h only. The substitution for x^n from (27) gives

$$\begin{aligned}\tfrac{1}{2}\varphi(h+x)=&F_0\left\{\varphi+\frac{2!}{(1!)^3}\,\frac{\varphi'}{4}+\frac{4!}{(2!)^3}\,\frac{\varphi''}{4^2}+\frac{6!}{(3!)^3}\,\frac{\varphi'''}{4^3}+\ .\ .\ .\right\}\\ &-F_1\left\{\frac{2!}{0!1!2!}\,\frac{\varphi'}{4}+\frac{4!}{1!2!3!}\,\frac{\varphi''}{4^2}+\frac{6!}{2!3!4!}\,\frac{\varphi'''}{4^3}+\ .\ .\ .\right\}\\ &+F_2\left\{\frac{4!}{0!2!4!}\,\frac{\varphi''}{4^2}+\frac{6!}{1!3!5!}\,\frac{\varphi'''}{4^3}+\ .\ .\ .\right\}\\ &-F_3\left\{\frac{6!}{0!3!6!}\,\frac{\varphi'''}{4^3}+\frac{8!}{1!4!7!}\,\frac{\varphi''''}{4^4}+\ .\ .\ .\right\}.\quad .\ .\ (28)\\ &+\ .\ .\ .\end{aligned}$$

The coefficient of every F is finite if the Taylor series is convergent. The arrangement of the coefficients in this form may evidently be discontinued at any stage, and the Taylor expansion substituted. The corresponding formula for use between the limits $x=\pm1$ may be written down at once. For the hypergeometric function of equation (11) terminates when n is the half of any positive integer, and therefore if $\frac{n}{2}$ is written for n

this formula includes both (16) and (17). The alterations required in (28) are now obvious.

The analogy between (28) and Fourier's series should be noted. If x is replaced by suitable circular functions the F's represent other circular functions. Thus

$$F\left(n, -n, \tfrac{1}{2}, \frac{1-\cos\theta}{2}\right)=(-)^n F\left(n, -n, \tfrac{1}{2}, \frac{1+\cos\theta}{2}\right)=\cos n\theta.$$

If n is even

$$F\left(\frac{n}{2}, -\frac{n}{2}, \tfrac{1}{2}, \sin^2\theta\right)=(-)^{\frac{n}{2}} F\left(\frac{n}{2}, -\frac{n}{2}, \tfrac{1}{2}, \cos^2\theta\right)=\cos n\theta,$$

and if n is odd

$$n \sin\theta\, F\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2}, \sin^2\theta\right)=\sin n\theta.$$

These relations indicate that the series coefficients in (28) may be replaced by integrals when summed to infinity. They may be so expressed in many different forms. The parallel shows that as with Taylor's series so with Fourier's series, a closer approximation when a finite number of terms only is retained may be obtained by modifying the coefficients from the values given by integration. The closest possible approximation is obtained when the end errors and all the maximum and minimum errors are numerically equal. The coefficients may be corrected in a similar manner to that employed for Taylor's series. The correcting term in this case may be written

$$\frac{1}{n+1}\Sigma(-)^\lambda \epsilon_\lambda (F_n f_0+F_{n-1} f_1+F_{n-2} f_2+ \ . \ . \ . +F_0 f_n),$$

where f is written for F when x is replaced by x_λ, or

$$\frac{1}{n+1}\Sigma\epsilon_\lambda (F_0 f_0+F_1 f_1+F_2 f_2+ \ . \ . \ . +F_n f_n),$$

the halving factor applying to extreme terms. The parallel with the ordinary Fourier procedure is evident.

When the function can be calculated simply by a direct method, and it is only intended to fit a given kind of formula to it, the ϵ's may be the actual values of the function at the stationary points. The same result is obtained by retaining a suitable number of terms in equation 28. Only those terms of the numerical coefficients corresponding to the terms neglected in the Taylor expansion need be calculated, the corrections alone being thus evaluated.

ABSTRACT.

In calculating functions from Taylor expansions or otherwise, the results obtained by summing any finite number of terms will differ to a greater or less extent from the true results. It is shown in the Paper that by suitable modifications of the coefficients the results obtained, even when comparatively few terms of the expansion are taken, can be made to approximate very closely to the true results for all values of the variable between selected limits.

DISCUSSION.

Mr. F. J. W. WHIPPLE asked why, when terms of higher order than n were to be neglected, the author found it advantageous to use $n+2$ points rather than $n+1$.

Mr. SMITH said it could be shown that the maximum accuracy was obtainable by taking $n+2$ points and giving equal and opposite errors at alternate points. He added that it was possible by the method of the Paper to represent a function which it was impossible to expand. As an example he mentioned the common logarithms from 1 to 10. Any ordinary expansion would be divergent over this range.

XVII. *Notes on a Method of Testing Bars of Magnet Steel.*

By N. W. McLachlan, *D.Sc.* (*Eng.*), *Member I.E.E.*

Received November 14, 1919.

Introduction.

The present Paper is an outline of a series of tests on magnet steels carried out several years ago. Ewing's double permeameter method* for round bars was adopted, a modification being introduced in the fitting of the yokes to the bars. Although this method gives results which are sufficiently accurate for most practical requirements, it is more laborious and less accurate than that developed by Messrs. Campbell and Dye,† in which differential search coils are used to measure the value of the magnetising force. The latter method possesses the advantage that the value of the flux density and the corresponding value of the magnetising force can be measured at any part of the bar. In Ewing's method the value of *B* is not uniform along the bar, owing to leakage between the bars in the two limbs of the permeameter. Since leakage occurs with both the long and the short permeameters, there is a certain amount of compensation, when the *B-H* curve is obtained by the method outlined by Ewing, as will be shown later when dealing with the correction to be applied for leakage. Moreover, the value of *B* being found experimentally with a search coil at the centre of the bar (where the leakage is small), and that of *H* (at the centre), calculated from the permeameter constants, it follows that, provided the leakage effect was the same for both permeameters, the values of *B* and *H* would be correct. The leakage effect is not the same for both permeameters, and the value of *H* as found by calculation is in error. The error for any of the bars tested, which were of low permeability, does not exceed 1 per cent. As a method of precision the above has little to recommend it, whilst the additional labour required in taking two sets of readings, combined with the fact that the final result is only found after reduction from two *B-H* curves, is such as to make it inferior to the differential coil method.

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* "Magnetic Induction in Iron and other Metals," "The Electrician," Vol. XXXVIII., p. 110, 1896.

† "Journal," I.E.E., Vol. LIV., p. 35, 1915.

Description of Apparatus.

A plan and elevation of the long permeameter is shown in Fig. 1. The short permeameter is the same in every respect as that in Fig. 1, excepting that the length of the winding is 3½ in. instead of 7. Six layers of 324 series turns are wound on each limb, the maximum value of H obtainable being 450 C.G.S. with a current of 20 amperes. The coils are wound on brass tubes fitted with end plates, split longitudinally to minimise eddy current effects. The space in a tube due to the removal of metal was filled with insulation to prevent the ends being drawn together when the coils were being wound. Although no deleterious effects were observed as a result of

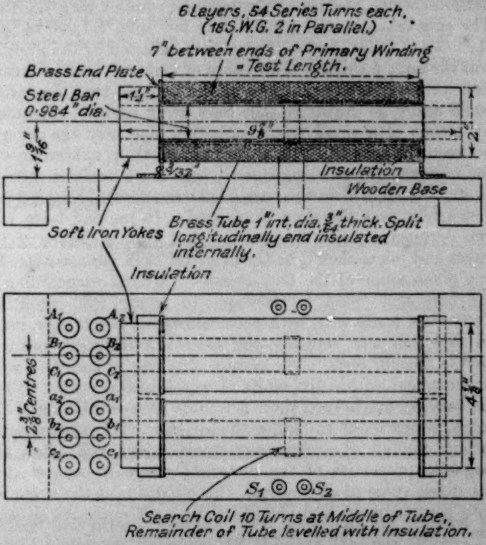
6 Layers, 54 Series Turns each.
(18 S.W.G. 2 in Parallel.)
7" between ends of Primary Winding = Test Length.
Brass End Plate
Steel Bar 0·984" dia.
1¼"
9⅞"
2"
1 9/16"
3/32"
Insulation
Wooden Base
Soft Iron Yokes
Brass Tube 1" int. dia. 3/64" thick. Split longitudinally and insulated internally.
Insulation
A1
A2
B1
B2
C1
C2
a2
a1
b2
b1
c2
c1
2⅜" Centres
4⅛"
S1
S2
Search Coil 10 Turns at Middle of Tube, Remainder of Tube levelled with Insulation.

FIG. 1.—PLAN AND ELEVATION OF LONG PERMEAMETER P_1.

A_1A_2=Terminals of first layer. 54 turns ($H=3{\cdot}82I$).
B_1B_2=Terminals of second layer. 108 turns.
C_1C_2=Terminals of third layer. 162 turns (H total$=22{\cdot}92I$), where I=current.

the use of brass, it is better to use an insulating material on which to wind the coils. Such a material is not likely to be so permanent, of course. The terminals are arranged so that one or more layers can be used, and in this way small values of *H* are obtained without using a very small current.

The bars tested were nominally 1 in. diameter as taken from the rolls. These were ground down by means of a precision grinding machine, to 0·984 in. diameter, thus removing the material which had been in immediate contact with the rolls. The cuts taken by the grinder were less than 1/1,000 in., and the mass of metal, beneath the surface, affected by the grinding was the least possible. Two yokes of very soft and permeable iron were accurately bored to fit the bars, so that the former could easily be drawn on (the bars being slightly lubricated) by means of a small screw jack. In this way the air gap between the bars and yokes was reduced to a minimum, and the necessity for pinching screws, whereby a variable amount of stress is applied to the bars when tested with different permeameters, was therefore eliminated. In making tests with two permeameters, it is essential that the reluctance of the yokes and joints should be the same for each permeameter for any given value of the flux density. Accurate machining of the bars and yokes is necessary to secure this condition. Tests were made with both permeameters by putting a mark on each bar and rotating the bar to a different position for the various tests. The results obtained in this way were in agreement within the limits of experimental error. When the bars are magnetised there is a force between them and the yokes, which tends to reduce the air gap. This introduces a certain amount of compressive stress, which will affect the magnetic properties of the materials where it occurs. For any given value of *B*, however, this condition is the same for both permeameters and the effect does not make its appearance in the corrected *B-H* curve.

The apparatus required to obtain *B-H* curves and hysteresis loops is shown diagrammatically in Fig. 2. The ammeter *A* could be used to read from 1/1,000th to 20 amperes, using four different shunts all of which were enclosed within the instrument case. The *B-H* curves were obtained by the method of reversals using a moving-coil ballistic galvanometer having a free period of 12 seconds. The calibration of the galvanometer was effected by means of a standard solenoid about 2 metres long, having a search coil of 400 turns, situated at its centre, and giving up to 5×10^6 interlinkages, *i.e.*, line turns, on reversal of the current. The search coil of the solenoid was permanently connected in the galvanometer circuit, so that the conditions under which the *B-H* curves were taken were identical with those when the galvanometer was calibrated, except that the iron was removed from the permeameter for the latter operation. In order to destroy any residual magnetism in the yokes, each yoke was subjected to a few sharp blows before being fitted to the bars.

It was thought that errors might be introduced owing to time lag of the flux in the interior of bars nearly 1 in. diameter,

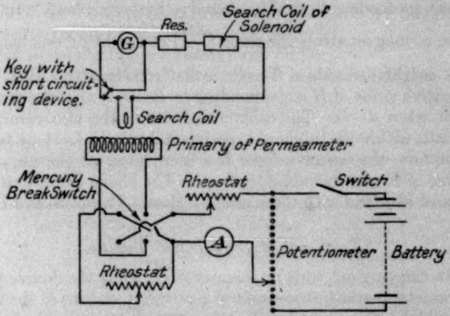
Search Coil of Solenoid
Res.
G
Key with short circuiting device.
Search Coil
Primary of Permeameter
Mercury BreakSwitch
Rheostat
Switch
A
Potentiometer
Battery
Rheostat

FIG. 2.—DIAGRAMMATIC SKETCH OF APPARATUS FOR OBTAINING *B-H* CURVES AND HYSTERESIS LOOPS.

thereby giving a smaller galvanometer throw than if the flux change had been transient. This point was investigated by using a Grassot fluxmeter to measure the flux change on reversal of various values of the current and comparing the results with those obtained with the galvanometer under identical conditions. The agreement between the results found by the two methods was accurate within the limits of errors of observation, although the flux change, especially on the steep portion of the *B-H* curve, with bars of relatively large permeability, was not complete before the galvanometer coil began to move (*see* Appendix).

Meaning of Symbols Used.

B =flux density in lines per square centimetre.

$B_{\text{rem.}}$=Remanence, *i.e.*, the value of B on the hysteresis loop for which $H=O$.

J =Intensity of magnetisation$=(B\text{-}H)/4\pi$.

$J_{\text{rem.}}$ =Remanent intensity of magnetisation on the hysteresis loop for which $H=O$.

$=B_{\text{rem.}}/4\pi$.

H=Magnetising force in C.G.S. units.

H_c=Coercive force, *i.e.*, the negative value of H required to reduce the remanent intensity of magnetisation $J_{\text{rem.}}$ to zero, when proceeding from $J_{\text{max.}}$ round a hysteresis loop. In the case of magnet steels the slope $\left(\frac{dB}{dH}\right)$ of the hysteresis loop in the neighbourhood of $B=O$, is sufficiently steep that the negative value of H corresponding to $B=O$, is almost equal to that when $J=O$. The difference between the two values of H falls within the limits of experimental error. In these tests, therefore, the coercive force has been taken as the negative value of H corresponding to $B=O$. The above definitions are almost identical with those used by Messrs. Campbell and Dye.

Treatment of Bars Before Testing.

In carrying out tests on magnet steels using the double permeameter method, it is essential that the treatment of the bars should be the same before being tested by individual permeameters, *i.e.*, the bars should have the same magnetic history. Each bar was, therefore, placed in a solenoid about 15 in. long, through the winding of which an alternating current of from 1 to 2 periods per second was passed for about 1 minute and then gradually reduced to zero. The condition of the bars was tested by slipping off a coil and observing the throw on a ballistic galvanometer connected to it. By this means it was possible to reduce the remanence ($B_{\text{rem.}}$) below 30 lines per square centimetre. To secure reliable results, the magnetic qualities of each of the pair of bars should be in close agreement, and in the experiments herein described, the permeability of individual bars of each pair differed by less than 1·5 per cent. By connecting the search coils on the two limbs of the permeameter in series, an average of the two bars is obtained.

Leakage.

Both permeameters were tested for magnetic leakage by winding a search coil, having the same number of turns as that on one of the brass tubes, on each yoke. The leakage value of B was ascertained by connecting the search coil on a yoke in opposition to that on one of the brass tubes, raising the sensitivity of the galvanometer by cutting resistance out of the secondary circuit and noting the throw on reversal of the current. Assuming that the rate of change of flux $\left(\frac{dB}{dt}\right)$ is the same through each coil, the throw is a measure of the flux which does not pass through the centre of the yoke, *i.e.*, the leakage flux.* The leakage can also be found by observing separate galvanometer throws with each coil.

Let δ_1=deflection with search coil on bar.

Let δ_2=difference in deflection with search coil on bar and that on yoke; then leakage fraction$=\frac{\delta_2}{\delta_1}=\lambda$.

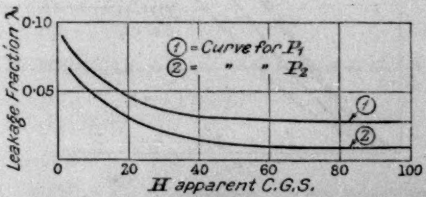

FIG. 3.—DIAGRAM SHOWING LEAKAGE FRACTION PLOTTED AGAINST APPARENT VALUES OF H, FOR BOTH PERMEAMETERS.

The circuits followed by leakage lines and lines through the yoke are in parallel; thus the same value of H produces the lines in both circuits. Since P_1 (long permeameter) is twice as long as P_2 (short permeameter), the leakage will be greater with the former than with the latter. This is borne out by experimental results, as illustrated in Fig. 3.

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* The value of B can be made almost uniform throughout the whole magnetic circuit by means of compensating coils at the ends of the permeameter limbs. The magnetising force due to these coils, however, affects the value of H at the centre of the bars by an amount which cannot be accurately calculated. The use of compensating coils on both permeameters would modify the two sets of results making it impossible for the true values of H to be easily and accurately found, since the test length with P_1 is twice that with P_2.

If the correction applied to find the "true" value of H at the middle of the bar is comparatively large, the effect of leakage may be appreciable. As the leakage is different for each permeameter, the reluctance of the paths (including leakage paths) other than the test lengths will also be different. Thus the additional ampere turns necessary to overcome this reluctance will not be the same for each permeameter, and the corrected value of H obtained from the curves, as shown in Fig. 7, will not be the "true" value of H. The error from this cause is most liable to occur on the steep portion of the B-H curve, where the correction for the effect of the yokes and air-gaps is large compared with the true value of H. It will be seen from Fig. 3 that the leakage is greatest as low values of H, and therefore at low flux densities. This is due to the fact that the yokes are being worked at very low permeability. It would not be advantageous to decrease the cross-sectional area of the yokes, thereby working them at higher permeability, since this would necessitate large corrections for high values of B, an increase in the leakage and a diminution in the largest value of the corrected magnetising force obtained with the permeameters.

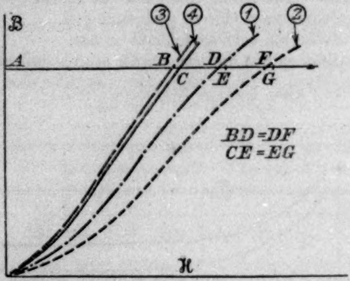

Fig. 4.

(1) Curve obtained with long permeameter.
(2) Curve obtained with short permeameter.
(3) Curve obtained from (1) and (2) by setting back DF from D.
(4) Curve obtained with zero leakage, *i.e.*, the true B-H curve. E and G are points in which curves (1) and (2) would cut AN for zero leakage.
BC = correction to be applied to (3) to obtain (4).

Owing to the complex nature of the leakage, it is difficult to estimate its effect on the "true" value of H to any degree of precision. In the following an attempt has been made to calculate approximately the correction to be applied in order to obtain the true value of H.

Correction for Leakage.

If it be assumed that B is proportional to H for the portions of the circuit other than the test lengths, a leakage fraction entails a corresponding diminution in the magnetising force necessary to overcome the reluctance of these portions, due to the decrease in B through them. If the true value of this force, *i.e.*, for zero leakage* is H, the actual value required with the permeameter is $H(1-\lambda)$. Thus the distance DF between curves (1) and (2) is in error by an amount depending, among other things, on λ_1 and λ_2, these being the leakage fractions with P_1 and P_2 respectively.

An approximate correction to be applied to curve (3), [obtained from (1) and (2)] in order to find the true curve (4), can be derived as follows :—

$$CD=CE(1-\lambda_1)$$

$$CF=2CE(1-\lambda_2) \qquad \therefore \quad CF=2CD\,\frac{(1-\lambda_2)}{1-\lambda_1}$$

$$BC=DF-CD, \text{ since } BD=DF;$$

$$=CF-2CD$$

$$=2CD\left(\frac{\lambda_1-\lambda_2}{1-\lambda_1}\right)$$

$$=2BD\left(\frac{\lambda_1-\lambda_2}{1+\lambda_1-2\lambda_2}\right) \text{ substituting } BD-BC \text{ for } CD;$$

$=2BD(\lambda_1-\lambda_2)$ approx., since $(\lambda_1-2\lambda_2)$ is small, compared with unity.

Now $AC \doteqdot AB$, and hence the percentage error on the true value of $H \doteqdot \dfrac{100(\lambda_1-\lambda_2)}{a}$, where $AB=2aBD$, and $\lambda_1\lambda_2$ are the leakage fractions for the apparent values of H (corresponding to the value of B under consideration) found with P_1 and P_2. It is of interest to observe that when $\lambda_1=\lambda_2$ the error vanishes. Using the above formula and the curves shown in Fig. 3, the error in the corrected value of H (as found by the method shown in Fig. 7) for the bars used, does not exceed 1 per cent. at any point on the B-H curve. As it is impossible to produce magnet steel in which the magnetic qualities *always* agree within 1 per cent., this error can be disregarded.

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* This, of course, can never be realised in practice.

B-H Curves, Hysteresis Loops, Remanence and Coercive Force.

After carefully demagnetising a pair of bars, as explained above, the B-H curve was taken with P_1. The bars were then demagnetised in a similar manner and the B-H curve taken with P_2. Having obtained these B-H curves, the corrected B-H curve was found by setting back the horizontal distance between the curves, so that AB is equal to BC, as shown in Fig. 7. This curve requires further correction, since the area of the search coil on the brass tube is 1·4 times the area of the bar. It is necessary, therefore, to substract 0·4 H from the ordinates of the curve, H being the value read on the curve found by the above procedure. Messrs. Campbell and Dye* substract this amount, automatically using a mutual inductance, the primary of which carries the current through the permeameter winding (or a fraction of it, using a shunt), the secondary being in series with the search coil. This procedure was

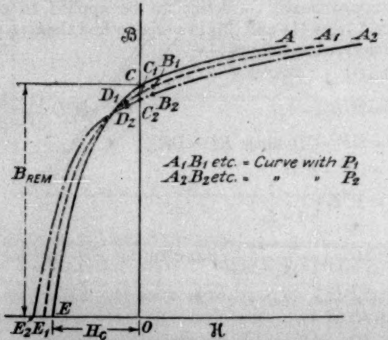

FIG. 5.—DIAGRAM ILLUSTRATING THE METHOD OF FINDING H'_c AND $B_{rem.}$

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* *Loc. cit.*

inapplicable in the present case because the current through the primary of the permeameter is proportional to the "apparent" H and not to the corrected H.

In order to get the corrected value of $H_{\text{max.}}=100, 200$ and 400 for the hysteresis loops, the "apparent" values of H required for P_1 and P_2 were taken from the B-H curves. Complete hysteresis loops were not taken for $H_{\text{max.}}=200$ and 400, but five points, viz., A, B, C, D and E, were determined with each permeameter (Fig. 5). A corresponds to the maximum value of H, B and D to points near the B axis, *i.e.*, near the point $B_{\text{rem.}}$, while C corresponds to the "apparent" remanence. The corrected value of $B_{\text{rem.}}$ can then be found quite accurately as shown in Fig. 5, by plotting the points B, C, D for each permeameter, and setting back the difference between the curves. The point in which the corrected curve cuts the B axis is the corrected value of $B_{\text{rem.}}$. Since the points are almost collinear $B_{\text{rem.}}$ may be found by calculation, using the co-ordinates of the points B and D. The equations to the lines through the respective pairs of points are given in Fig. 6. If m_1 and m_2 are the slopes of these lines, we have

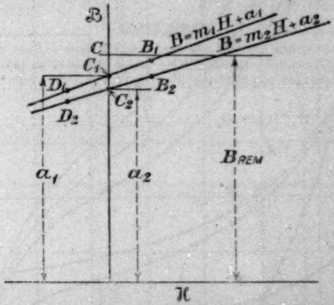

FIG. 6.—DIAGRAM ILLUSTRATING THE METHOD OF CALCULATING $B_{\text{rem.}}$.

$$B_{\text{rem.}}=\frac{2m_1A_2-m_2A_1}{2m_1-m_2}$$

H_c^1 is found by subtracting E_1E_2 from OE_1 (*see* Fig. 5).

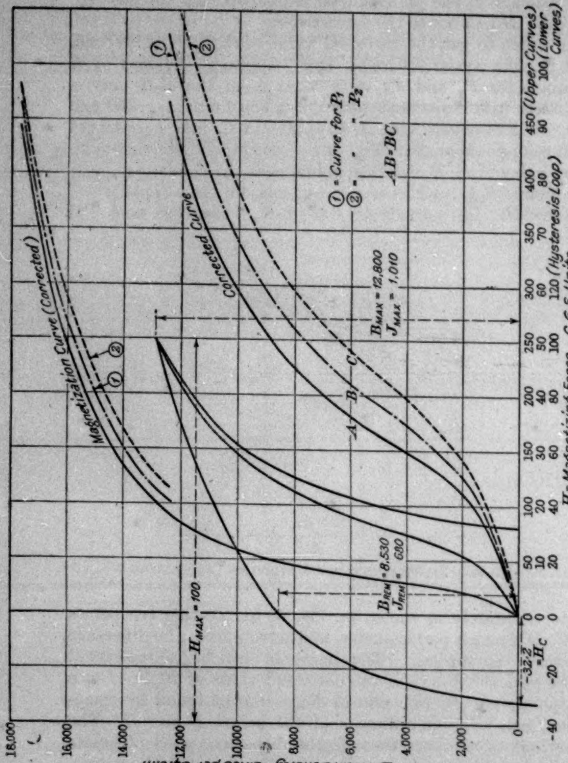

FIG. 7.—DIAGRAM SHOWING *B-H* CURVE AND HYSTERESIS LOOP OBTAINED WITH TWO SIMILAR BARS OF MAGNET STEEL.

In conclusion, the author wishes to convey his best thanks to Prof. E. W. Marchant, D.Sc., and to the Council of the Liverpool University for the facilities provided to enable the work to be carried out.

ABSTRACT.

In the Paper tests on cylindrical bars of magnet steel 1 in. diameter and 10 in. long are described. The tests were conducted using a slight modification of Ewing's double permeameter method. Instead of employing pinching screws in the yokes, the bars and yokes were ground accurately to 1/1,000th in., and arranged to make a good push fit. The method is compared with that in which use is made of differential coils for measuring the value of the magnetising force *in situ*, as developed at the National Physical Laboratory. It is shown that tests with the latter method can be conducted more speedily and accurately than with the double permeameter method. The variation in the magnetising force along the bar and the leakage between the pairs of bars in the permeameter is treated. A formula is developed, by means of which, with the aid of experimental data given in the Paper, a correction can be applied to allow for leakage effects. A *B-H* curve and hysteresis loop are given for a certain sample of magnet steel; also the details of the permeameter.

DISCUSSION.

Dr. D. Owen said it was evident that the leakage did not occur only at the yokes, but began near the centre of the bars. The applied field would therefore be strongest at the centre, so that taking the average field, as done by Ewing, would give too low a value.

XVIII. *On the Forces Acting on Heated Metal Foil Surfaces in Rarefied Gases.* *By* GILBERT D. WEST, *M.Sc.* (*Lond.*).

RECEIVED DECEMBER 18, 1919.

Arrangement of Paper.

(i.) Introduction.
(ii.) Apparatus.
(iii.) Experimental results and their reduction.
(iv.) Theoretical discussion of results.

Introduction.

THE present research arises out of two previous Papers to the Physical Society.* Both were attempts to measure the pressure of light by a very simple method. A strip of gold leaf was suspended vertically in a test-tube, and, on exposure to radiation, the strip was deflected through a small angle. A measurement of this deflection was made by a microscope, and when the weight of the strip was known, the radiation pressure could be calculated.

It was thought that the customary "radiometer effects" by which most light-pressure measurement are disturbed, would here be very small, owing to the almost negligible difference in temperature of the two surfaces of the strip. Experiment showed, however, that although the ordinary radiometer effects were apparently absent, new gas action effects appeared. It was established that symmetrical placing of the strip had much to do with their elimination, and, at certain gas pressures, reasonably accurate measurements of the pressure of light were possible. Nevertheless, in spite of this, a separate investigation of such effects was felt to be necessary. Chiefly with a view, therefore, to the elimination of troublesome sources of error, the present work was started.

It has since been found, however, that the nature of the gas action in itself is interesting, and that, as the whole problem of the equilibrium conditions in a rarefied gas is involved, offers a wide scope for research. A considerable amount of such research has already been done. Thus, in addition to the classic experiments of Crookes, there is much careful work by the Danish physicist Knudsen.† Moreover, a number of physical instruments, such, for instance, as the pressure gauges

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* "Proc." Phys. Soc., XXV., p. 324, 1913, and XXVIII., p. 259, 1916.
† "Ann. d. Phys.," 1910 and 1911.

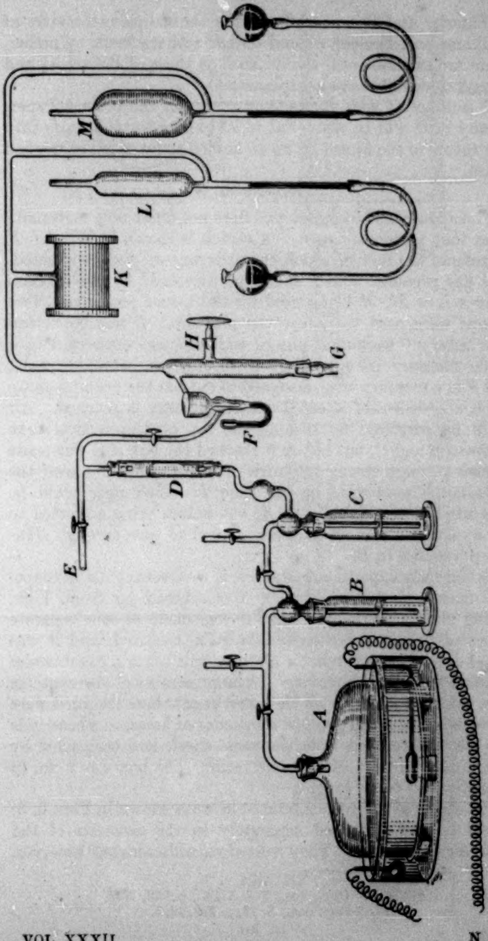
M
L
K
H
G
F
E
D
C
B
A

Fig. 1.

of Shrader and Sherwood,* and the thermo-galvanometers of Williams and Eccles† depend on gas action effects. Further, there are experimental results, such as those of Bottomley and King,‡ which still await explanation.

A number of wide issues thus arise, but the present Paper, at any rate, will be restricted to an experimental inquiry into the nature of the forces acting on heated strips of foil in rarefied gases.

Apparatus.

The apparatus here employed does not differ very materially from that previously used. A sketch is shown in Fig. 1. *K* represents the box in which the experiments were conducted. The gas pressure within it can be measured by the McLeod gauges *L* or *M*, *M* being used for the lowest pressures. Two pumps were used to reduce the pressure. *E* was connected to a rotary "backing" pump, whilst *G* was connected to a Gæde mercury pump. *H* is a phosphorus pentoxide bulb, and *F* is a mercury trap, designed to cut out the backing pump when a pressure of about 1 cm. of mercury is reached. Air could be admitted by turning the tap on the vertical tube between *B* and *C*, but before it reached the box *K* it was made to pass through strong sulphuric acid in *C*, and then over the phosphorus pentoxide in the tube *D*. Hydrogen could be generated electrolytically at *A*, but before being admitted to the apparatus, was, in addition, forced to pass through alkaline pyrogallol in *B*.

In the early experiments the box *K* was rectangular in shape, and measured 24 cm. long by 8 cm. broad by 8 cm. high. Owing chiefly to the fact that it was made of four separate pieces of brass, however, minute leaks occurred, and it was found difficult to get reliable observations below a gas pressure of about 0·0005 cm. mercury. A large number of observations were taken, but nearly all the experiments here recorded were made in a box constructed of a cylinder of brass, to whose ends two glass plates, cut from the same sheet, had been fixed by means of a mixture of wax and resin. The box was 9 cm. in diameter and 6 cm. long.

The strips of foil were mounted in ways shown in Figs. 2, 3, 4 and 5, and described separately in the accounts of the various experiments. They were always illuminated, however,

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* "Phys. Rev.," XII., 1918, p. 70.

† "Proc." Phys. Soc., Vol. XXX., p. 253, 1918.

‡ "Proc." Roy. Soc., A. 79, p. 285, 1907.

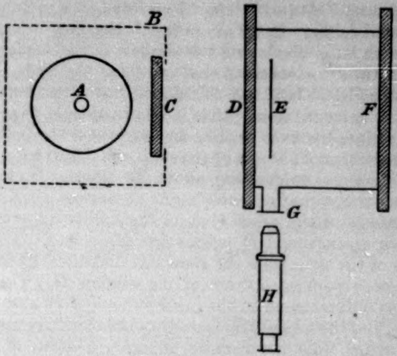

FIG. 2.

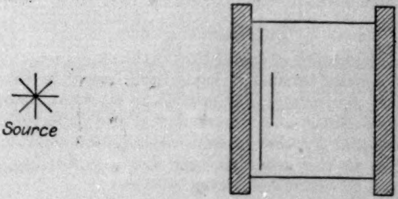

FIG. 3.

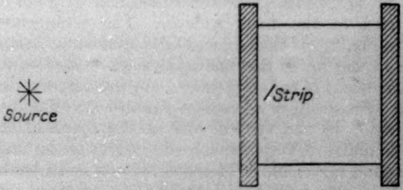
Source
Strip

FIG. 4.

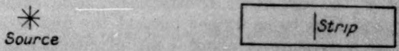
Source
Strip

FIG. 5.

by a 500 watt "Atmos" lamp *A* enclosed in a metal box *B*, represented in Fig. 2. The walls of the box were thickly coated with lamp black, and the window *C* was made of glass cut from the same sheet as that used for the ends *D* and *F* of the cylindrical box. A calculation based on the shape of the metal filament, showed that it was justifiable to apply the inverse square law even as close as the glass of the bulb.

The deflection of the end of the strip was observed by means of a microscope magnifying about 30 times. In the later experiments observations were made through a small window *G* in the side of the vessel, but in the earlier experiments a periscopic arrangement of prisms served instead. The mean distance of the strip from the glass was measured by means of a microscope working in front of the window *D*. The microscope was first focussed on the inner surface of *D*, and then on the strip, and the forward movement of the objective measured. As the strips were never quite straight, a mean of several readings was taken. With the strip in front of the screen, as in Experiment VI., no such accurate measurement was possible.

Experimental Results.

The present order of description of the experiments has been adopted, chiefly because it lends itself easily to subsequent theoretical treatment. The order is in no sense historic.

It should further be remarked that the deflections of the strip produced by the pressure of radiation only introduce themselves in this work as very small corrections, and in general can be entirely neglected.

Experiment I.—Strips of foil, of various widths, were suspended by horizontal glass fibres attached to glass distance pieces fixed to the front window. The arrangement was similar to Fig. 2. If the box was filled with air at atmospheric pressure, it was found that the incidence of radiation caused a slight movement of the strip away from the nearest glass wall, followed by a stronger movement towards it. Small changes in inclination to the vertical and in the shape of the strip seemed to make very appreciable differences to the magnitude of the second movement. Moreover, a very considerable time elapsed before the maximum deflection of the strip was reached. As the gas pressure was lowered the movement became less and less, but at the same time it established itself rather more quickly, and thus to an extent masked the first movement, Finally, at a pressure of about 1 cm. of mercury there was hardly any movement at all. With hydrogen gas, even at atmospheric pressure, only very slight movements were observed.

For reasons given in the section on the theoretical discussion of results, it is supposed that the second movement referred to above was caused by the establishment of convection currents. The first movement is most probably connected with the type of movement that predominates at pressures below 1 cm. of mercury. Further experiments refer to such pressures.

Experiment II.—The extreme thinness of the foils used would in itself suggest that the movements of the strips were independent of the difference of temperature of their surfaces. To make sure, however, the lamp was placed first one side of the box and then on the other. Examples of deflections obtained in this way are shown in Figs. 6 and 7.* It will be seen that with increasing rarefaction, the deflection of the strip rises, reaches a maximum, and finally falls. Nevertheless throughout it will be noticed that on whichever side the lamp is placed, the deflections are approximately the same. Many other such readings were taken, but on the average, still no certain difference in the two sets could be detected.

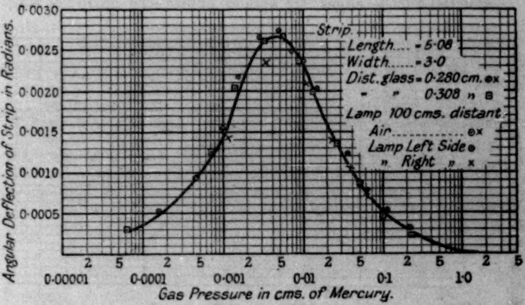

FIG. 6.—DEFLECTIONS OF COPPER STRIP NO. III. IN CYLINDRICAL BRASS BOX. Graphical Record of Results.

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* Semi-logarithmic paper has been used owing to the great range of the gas pressures.

Experiment III.—The previous experiment shows that the extremely small difference of temperature of the two surfaces at the strip is not instrumental in producing the deflections observed. Hence the forces acting on the strip must arise in a different way from those that produce a movement of the vanes in a Crookes radiometer, and it is natural to inquire if the pressure distribution is also different. On a Crookes vane, it is supposed that the excess pressure is restricted to a narrow margin near the edge, whilst the pressure at the centre is supposed to differ but little from the normal gas pressure.

Consider the case of two long strips, one narrower than the other, but both at the same distance from the glass wall, and also both 1°C. above its temperature. If the excess pressure is restricted to the edges, we should expect the narrower strip to be deflected most. If, however, the pressure is uniform, the deflections will be the same, whilst if the pressure increases from the edge to the centre, the larger strip will be deflected most.

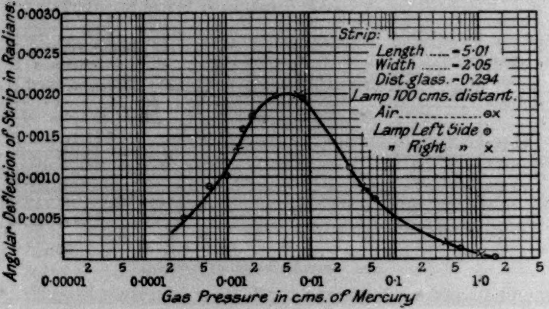

FIG. 7.—DEFLECTIONS OF COPPER STRIP NO. II. IN CYLINDRICAL BRASS BOX. Graphical Record of Results.

In the present experiment, two copper foil strips, widths 2 cm. and 3 cm. respectively, were placed at a distance of 0·3 cm. from the glass wall of the box. The distance 0·3 cm. could be considered small compared to the width of either strip, and thus at a given pressure, the temperature rises of the strips under the influence of radiation from a lamp a definite distance away, could be considered approximately the same.

The results are shown in Figs. 6 and 7. It will be seen that the wider strip is deflected most at all pressures. It is possible that a certain amount of this difference may be accounted for by the slightly higher rise of temperature of the wider strip,* but it is probable that the chief difference is accounted for by a pressure which rises in value from the edges, quickly attains a maximum value, and remains constant over the greater part of the strip.

Similarly corresponding curves were obtained when the box was filled with hydrogen.

Experiment IV.—It had been noticed that the deflection of the strip depended on its distance from the glass wall. With a view therefore to obtaining more definite information, a series of readings was taken of the deflections of the copper strip at about twice its previous distance. As shown by Fig. 8, all the deflections observed are smaller, but as the temperature to which the strip rises in this case is necessarily higher, the meaning of the present results in relation to the previous ones, requires to be made clear by a reduction of both sets to deflections per 1°C. rise of temperature of the strip. Such a reduction will be referred to later.

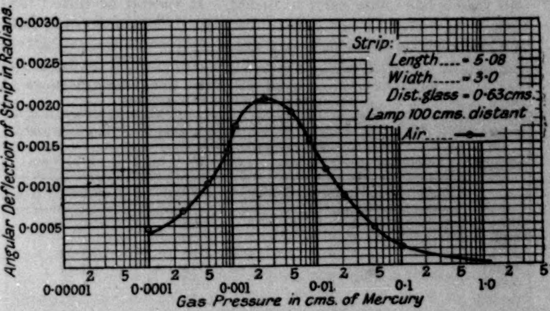

FIG. 8.—DEFLECTIONS OF COPPER STRIP III. IN CYLINDRICAL BRASS BOX. Graphical Record of Results.

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* The formulæ of Kirchhoff (Winkelmann, "Handbuch der Physik," IV., pp. 35-36) and Maxwell ("Elec. and Mag.," Vol. I., p. 286) are apparently not of much use in helping to give a better approximation to the true temperature rise.

Experiment V.—A larger amount of heat is conducted from the edges of the strip than from the central portions, and a temperature gradient, whose value depends on the thickness of the strip, thus exists. It was thought that this temperature gradient might have something to do with the production of the pressure on the strip, and with a view to varying the temperature gradient therefore, experiments were carried out with foils ranging in thickness from about 100×10^{-5} cm. to $0·9\times10^{-5}$ cm. In the former case a sufficient difference of temperature was produced between the two surfaces to give deflections that depended on the side on which the lamp was placed, to the extent of as much as 20 per cent. In the following table of results calculated from curves, the mean of these two values has alone been recorded. It should be noted further that, owing to their smallness, the deflections could not, in this case, be observed very accurately.

Gas pressure cm. mercury.	Ratios of forces on strips, calculated from strip deflections (lamp same distance throughout).		
	Gold strip.	Copper strip.	Aluminium strip.
	Thickness=$0·9\times10^{-5}$ cm.	$0·4\times10^{-4}$ cm.	$0·1\times10^{-2}$ cm.
	Thermal conductivity=0·70.	0·92	0·5
0·1	1	1·03	...
0·03	1	1·20	...
0·01	1	1·19	1·13
0·003	1	1·19	1·44
0·001	1	1·03	1·39
0·0003	1	0·98	1·37

It was thought probable that the differences in the forces acting on the strips could be satisfactorily accounted for by the different amounts of radiation absorbed, and hence by the different temperature rises. Measurements were accordingly made of the relative absorbing powers of gold, copper and aluminium foil, by observing, with a thermo-junction, the initial rate of rise of temperature on exposure to the radiation, of a copper block, one of whose faces had been covered with the foil in question. The method has certain disadvantages, for the absorbing power of a metal surface is very small, and care has to be taken to avoid heating of the block by warm air currents produced in the vessel, in which the block is contained. Moreover, the surface of the thin foil may suffer a change when it is stuck on the surface of the block. The following values, however, were obtained for the relative absorbing powers :—

Gold	=1.
Copper	=0·98.
Aluminium	=1·41.

It would seem therefore as if the differences in the forces acting on the various strips could be accounted for chiefly by the different amounts of radiation absorbed, and hence by the different temperature rises. The temperature gradient in the strip from the edge to the centre would, moreover, seem to have had but little influence in the production of the pressure on the strip.

Experiment VI.—It had been noticed that if the front glass of the box containing the lamp was removed, and the light switched on for a considerable period, the strip did not return to the original zero. By filtering the radiation through one or more plates of glass cut from the same sheet as the window of the box, however, it was possible to reduce the creeping of the zero to a very small value.

The phenomena connected with the repulsion of a metal strip from a hot wall appeared to be worth investigation, and accordingly the following experiment was planned :—

At a distance of about 0·6 cm. from the glass wall a large sheet of aluminium foil 0·001 cm. thick covering nearly the whole of the front face of the box was fixed by means of silk threads attached to small glass blocks. Attached in turn to the aluminium foil was a fine horizontally-placed fibre of glass or silk carrying the strip of aluminium foil. The first strip used was 4·6 cm. long and 3 cm. broad and $0{\cdot}67 \times 10^{-4}$ cm. thick, and was placed first 0·6 cm. from the aluminium screen and later 0·3 cm. The second strip used was of the same thickness, but was 2·5 cm. long and 1·0 cm. broad, and was placed approximately 0·1 cm. from the screen. The whole arrangement is represented in Fig. 3. Radiation from the lamp falls on the screen and raises its temperature, and, if the gas pressure is low enough, a deflection of the thin aluminium strip can be observed in the usual way. A series of results was thus obtained, but like those of Experiment IV., they are more easily considered when reduced so as to show the force per square centimetre of strip per 1°C. rise of temperature of the screen. The reduced results are shown in Fig. 10.

Experiment VII.—Instead of mounting the strips parallel to the glass wall, the effect was tried of mounting a strip 0·6 cm. wide at an angle of about 45°, the nearest edge of the strip being about 0·1 cm. from the glass. A diagram of the arrangement is given in Fig. 4. Another strip was similarly mounted at an angle of about 45° to the screen. When intense radiation was concentrated on the strip near the glass wall by means of a lens, it was found that, over a considerable range of pressure near 0·01 cm. mercury, the strip, instead of being deflected away from the glass wall, would sometimes move edgewise towards it, and remain there until the radiation was cut off. If the nearest edge of the strip was more than about 0·1 cm. from the glass wall a slight mechanical shock was generally necessary before the edgewise movement would take place. When the radiation was concentrated on the screen (the lamp being on the side remote from the strip) the strip was always repelled, and the edgewise movement could never be induced.

Experiment VIII.—A strip about 0·6 cm. wide and about 3·5 cm. long was placed symmetrically in a small glass box 1 cm. wide, 2·5 cm. long and 4 cm. high. A plan of the arrangement is shown in Fig. 5. When light from the lamp was concentrated on the strip by means of a lens, and a small mechanical shock administered to the table on which the box rested, the strip moved edgewise to one or other face of the box, and remained there until the radiation was cut off. The action was most vigorous at a pressure of about 0·01 cm. mercury, but extended over a considerable range.

Reduction of Curves.

The results of Experiments III., IV. and VI. can be rendered much clearer by reducing them to the force per square centimetre of strip or screen per 1°C. rise of its temperature. Before any such reduction can be made, however, we require to know the way in which the temperature of the strip or screen varies as the gas pressure is reduced—the distance of the lamp, of course, being kept constant.

Reduction of gas pressure diminishes the effective conductivity of the gas, and although at first the diminution is small, it becomes marked when the mean free path of the molecules reaches an appreciable fraction of the distance of the strip or screen to the glass wall. Thus strip or screen temperatures rise with reduction of gas pressure.

To measure the temperature rises of the strips, several experimental methods were tried, but eventually the thin strip of metal leaf was replaced by foil of sufficient thickness to admit of the attachment of a very fine thermo-junction. The method gave consistent results, but there was some doubt as to what extent they corresponded with the actual temperature rises of the original strips.

The problem was therefore approached from a different point of view. By coating the surface of a copper block with the foil in question, and by measuring the initial rate of rise of temperature when the surface was exposed to radiation from the lamp, an estimate of the quantity of energy Q absorbed per square centimetre per second by the strip was obtained. By equating Q to the heat lost from the strip by conduction through the gas and by radiation, the author found that, as a first approximation, T, the temperature rise of the strip could be written

$$T=Q\Big/\left[k\left\{\frac{1}{d_1+2c\lambda}+\frac{1}{d_2+2c\lambda}\right\}+2R\right],$$

where k is the conductivity of the gas, d_1 and d_2 the distances from the glass walls, R the emissivity, λ the mean free path of the molecules, and where c is a constant depending on the nature of the gas, and whose significance will be discussed later.

A correction was also applied for the conduction of heat through the supports of the strip.

Although the absolute values differ somewhat, the fractional variations of the temperature rises of the strips with gas pressure, obtained by this and by the thermo-junction method, are in good agreement as low as 0·01 cm. mercury, but after this gas pressure, the former values are markedly in excess. It is the values given by the calculation that have been used in the reduction of the curves, although it is admitted that uncertainties in the values of c and R may introduce serious errors.

So far as the immediate purpose of the present work is concerned, however, no very accurate results are required. It is rather qualitative guidance that is sought, and the reduced curves, which are shown in Figs. 9 and 10, should be regarded in this light.

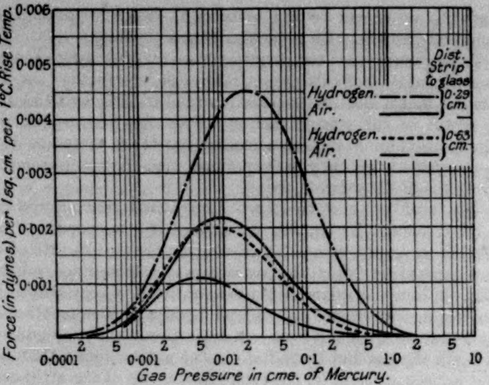

FIG. 9.—RELATION BETWEEN FORCE ON 1 SQ. CM. OF COPPER STRIP III. PER 1°C. RISE OF TEMPERATURE, AND GAS PRESSURE.

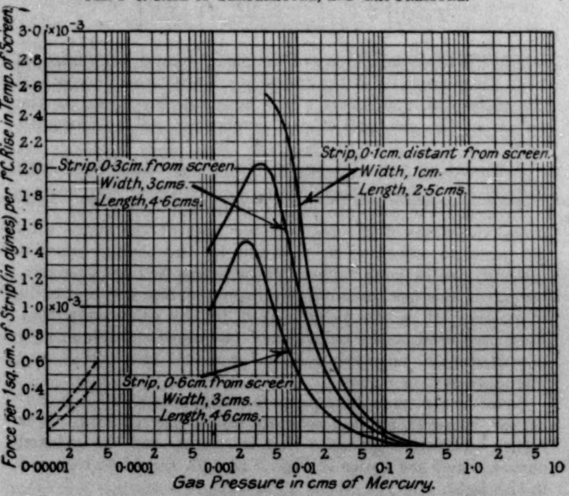

FIG. 10.—RELATION BETWEEN FORCE PER 1 SQ. CM. OF STRIP PER 1°C. RISE IN TEMPERATURE OF SCREEN, AND GAS PRESSURE. Aluminium Strips VIII. and IX.

Theoretical Discussion of Results.

It is natural to attribute the movements of a light, freely suspended, heated body, to convection currents in the surrounding gas. Convection currents, however, in any apparatus depend fundamentally on the ability of a small volume of air to maintain for some time its high temperature and hence its buoyancy. If the mean free path of the molecules be sufficiently large, it is clear that they will drive their way through any such small volume, and that a uniform condidition will soon be established. Thus, with reduction of pressure and increase of mean free path, convection currents tend to disappear.[*]

In Experiment I. it will be noticed that from atmospheric pressure down to about 1 cm. of mercury, two movements of the strip are clearly superposed. The more slowly established movement dies out as the pressure is reduced, whereas the quickly established movement increases with reduction of pressure. It has been thought reasonable in this, as in previous work,[†] to attribute the former movement to convection currents, whilst the latter movement has been attributed to forces about whose nature a theory is now to be put forward. The theory is based on a previous Paper by the author[‡] on thermal transpiration, and is apparently capable of explaining all the present experimental facts.

It will be remembered that in the Paper referred to it was shown that if a temperature gradient were maintained along a tube of uniform bore, the walls of the tube, in forcing the gas to take their temperature, exerted a traction upon the gas, and caused it to flow from the cold to the hot side. With finite spaces at the ends of the tube, it was shown also that a sufficient pressure was developed on the hot side to cause an equal flow, of the Poiseuille type, in the reverse direction.

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[*] In this connection it should be noted that rapidly moving molecules, such as those of hydrogen, are of necessity more effective in removing temperature inequalities than, for instance, the more slowly moving molecules of air. The difficulty of maintaining convection currents in hydrogen is now well known.

[†] "Proc." Phys. Soc., XXVIII., p. 259, 1916.

[‡] "Proc." Phys. Soc., XXXI., p. 278, 1919. As frequent references are made to this Paper, the author would like to take the opportunity of making the following errata :—

P. 281, fourth and fifth lines from bottom, for $-\frac{1}{2}\frac{d\Omega}{dx}\lambda$ read $+\frac{1}{2}\frac{d\Omega}{dx}\lambda$.

P. 285, fourteenth line from top, for $\frac{2k}{T}\frac{dT}{dx}$ read $\frac{k}{T}\frac{dT}{dx}$.

P. 287, ninth line from bottom, for 1 cm. read 0·1 cm.

The result of the superposition of the two flows was to give a gas current near the surface of the tube from the cold side to the hot side, and a current in the reverse direction along the axis, whilst between the two there was a surface of zero velocity. With the reduction of gas pressure the Poiseuille counter-flow became less and less, until at the highest rarefactions, it was quite unimportant, and the hot regions were enabled to maintain undiminished their higher pressures.

In this latter stage, the pressure temperature gradient was found proportional to the pressure of the gas, but independent of its nature. At high pressures, however, when the mean free path of the molecules was small compared to the diameter of the tube, it was found to be inversely proportional to the gas pressure, and to be greater for hydrogen than for air.

The theory was worked out on conditions that were much simpler than those that obtain here, and it is only proposed in the present instance to look to the theory for general guidance, and not for quantitative agreements. In this, as in the previous work, however, a flow of gas is supposed to take place from the cold to the hot regions, but as the extent of a "hot region" changes with variation of gas pressure, some preliminary considerations are necessary.

Consider first of all the case of two parallel plates—one hot and the other cold. We may regard the molecules that strike the hot surface not only as coming on an average from a distance λ, the mean free path of the molecules, but also as possessing the mean temperature of this region. If, after reflection at the surface, these molecules merely acquire its temperature, it is clear that the mean temperature of the surface layer of gas is necessarily lower than that of the surface itself. A surface temperature discontinuity thus exists, which becomes more marked the larger the value of λ.

One result of the discontinuity is that the quantity of heat passing between the plates is reduced, and it is now customary to regard this reduction as due to a change in the surface conditions, and not to a change in the internal conductivity of the gas itself. Smolan,* Warburg and Gehrcke,† and Lasareff‡ have in fact shown that the quantity of heat conducted per second between the two unit surfaces distant d apart and differing in temperature by 1°C. may be expressed as $\frac{k}{d+2c\lambda}$, where k is the ordinary conductivity of the gas. The effective distance between the plates must thus be increased by $2c\lambda$.

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* "Akad. Wiss. Wien. S. Ber.," CVIII., pp. 5-23, 1899.

† "Ann. d. Phys.," II., 1, pp. 102-114, 1900.

‡ "Ann. d. Phys.," XXXVII., 2, pp. 233-246, 1912.

If a molecule reached thermal equilibrium on collision, the value of c would be unity. Many observers besides those mentioned, and including Soddy and Berry,* Knudsen,† Smoluchowski,‡ and Langmuir,§ have demonstrated, however, that this is not the case, and have assigned values to c. Smoluchowski has pointed out that the interchange of energy between colliding spheres is the more imperfect the greater the difference of their masses, and we might thus expect the temperature discontinuity to be greater for the light than for the heavy gases. This is so, but Langmuir and Knudsen have shown that the whole problem is much modified by the presence of surface layers of condensed gas. Support is lent to their conclusions by the fact that at a given pressure, c varies but little with the nature of the surface, provided only that the latter is polished.

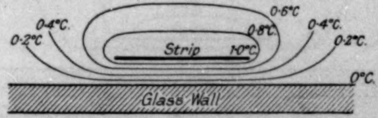

FIG. 11.—ISOTHEIMS. MEAN FREE PATH SMALL COMPARED WITH DISTANCE OF STRIP TO GLASS WALL.
Rise of Temperature of Strip=1°C.

According to Warburg, c varies somewhat with gas pressure, but for the present work it may be taken as constant and about 5·8 for hydrogen and 1·65 for air.

Let us now turn our attention to the case of the metal strip suspended parallel to the glass wall, as in Fig. 2. The shapes of the isothermals near the strip and the closest wall are, at high pressures, somewhat as represented in Fig. 11. As, however, the gas pressure is lowered, the high temperature isothermals tend to disappear as a result of the temperature discontinuity, and we are confronted with a system of isothermals roughly sketched in Fig. 12. Finally, when the mean free path is long compared to the dimensions of the vessel, there will be hardly any temperature gradient from the strip to the opposite plate. With infinite plates, the whole of the gas would be at a temperature half way between that of the hot and cold plates, but in the present instance the temperature will vary slightly from point to point. The temperature near the edges of the strip for instance, would be somewhat lower than near the middle of the strip.

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* "Proc.," Royal Soc. A LXXXIV., p. 576, 1911.
† "Ann. d. Phys.," XXXIV., 4, p. 593, 1911.
‡ "Phil. Mag.," XXI., p. 11, 1911.
§ "Phys. Rev.," Vol. II., No. 5, p. 329, 1913.

At medium pressures, therefore, any plane placed parallel to the strip and near to it, will coincide with the isothermals in the central portions, but will intersect successive isothermals in the regions near the edges. Similar considerations apply to a plane placed close to the nearest wall. We have, in fact, both in the gas layer at the edges of the strip and in the gas layer on the opposite wall, a definite temperature gradient. Although this temperature gradient is not uniform in character, and although it does not extend directly across from strip to plate, yet there is a certain resemblance to the conditions of the previous Paper,* and it has been thought a legitimate step to apply in a very general way the results there obtained.

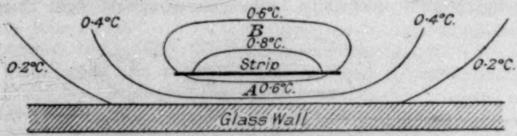

FIG. 12.—ISOTHERMS. PRESSURE SUCH THAT $c\lambda$=DISTANCE OF STRIP TO GLASS WALL.

Rise of Temperature of Strip=1°C.

If this plan be adopted, we must assume that on the two sides of the strip and on the opposite wall, we have a flow of gas from the exterior to the interior, together with an equal central counter-flow from the interior to the exterior. The superposition of these two flows might thus quite well be represented by a diagram such as Fig. 13. Over a wide region near *A* there will be little motion, and the pressure will be sensibly constant, whilst near the edges the pressure will fall off gradually—partly as the result of the existence of motion, and partly as the result of the flow on the other surface of the strip. This latter flow, of course, produces a pressure in opposition to that on the side A and when the strip is placed symmetrically in any vessel, the resultant pressure is in theory zero, and in practice negligible. In fact, strips mounted thus, were previously used for the estimation of the pressure of light.

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* "Proc." Phys. Soc., XXXI., p. 278, 1919.

It is thus possible to explain why in Experiment II. the deflections of the strip do not depend on a difference of temperature of the surfaces, and also why in Experiment III. the pressures do not appear as " edge effects."

The results of Experiment V. with thicker strips, likewise find an easy explanation in the present theory. We should expect the temperature gradient due to the temperature discontinuity, far to outshadow in importance the small temperature gradient on the strip itself, due to a greater quantity of heat being conducted away from its edges. It is the temperature of the layer of gas that is important, not that of the material of the strip.

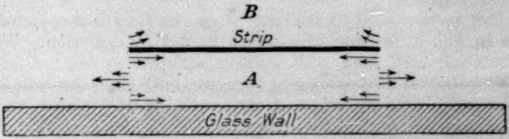

FIG. 13.—FLOW OF GAS ROUND HEATED STRIP PLACED IN FRONT OF COLD GLASS WALL.

In addition, it may be noted that the force per square centimetre on the strip is greater for hydrogen than for air. According to the previous theory, the ratio should be

$$\eta^2_h/\rho_h : \eta^2_a/\rho_a = 3{\cdot}4 : 1,$$

where η and ρ refer to viscosities and densities respectively. The ratios in both the experimental curves are roughly of this order, although no exact agreement can be expected.

Other points of agreement may be noted. The pressure at which a maximum value is obtained for the resultant force on the strip, is lower for air than for hydrogen, both according to theory and to experiment. Further, as in Experiment IV., the maximum value occurs at a lower pressure when the distance of the strip from the glass wall is greater. This corresponds in the previous work with the widening of the bore of the tube. It is easily seen that such a widening results in shifting the maximum of the curves to the left.

As we proceed to still lower pressures, questions of gas flow become far less important, and we are then concerned instead with the mean temperature of the two regions on either side of the plate. As their limiting values are very close, the resultant pressure on the strip soon begins to fall to a very low figure. It should be noted, moreover, that the values both for hydrogen and for air approach each other, and that the pressure on the strip is now roughly proportional (whereas before it was inversely proportional), to the gas pressure. This is all in accord with the conclusions of the previous Paper.

Let us now pass to the case where the strip is suspended, as in Fig. 3, in front of a screen heated by radiation. We are confronted here with quite a different system of isothermals. At high pressures they are, with the exception of the region near the edges of the screen, parallel planes, and the strip will acquire the temperature of the isothermal with which it coincides. We must, however, associate with the rise of temperature of the screen, a flow system similar to that round the strip in Fig. 13. Notwithstanding this a strip suspended parallel to the screen as in Fig. 3 will be out of the region in which the flow occurs, and it will not be affected. An inspection of the curves of Fig. 9 reveals the fact that at a gas pressure of 0·1 cm., for instance, there must, by analogy with the case of the strip, be a very considerable gas flow round the edges of the screen. Nevertheless the deflections of the strip are very small, as shown in Fig. 10.

With decrease of gas pressure, the previously inconsiderable temperature discontinuities at the surfaces, have to be taken into account, together with the resultant changes of form of the isothermals. Let us first suppose the screen to be a sufficiently good conductor to maintain itself at a uniform temperature. Then, in addition to the temperature discontinuities at the screen and at the glass wall, there will be others at the surfaces of the strip, and thus in effect the distance from the screen to the opposite cold wall will be increased by $2c\lambda$. A somewhat smaller quantity of heat thus passes from the portion of the screen opposite the strip, and the temperature gradient is likewise smaller. An attempt has been made to embody these facts in the rough drawing of the isotherms given in Fig. 14.

As the pressure is still further reduced, region A will approximate to a mean temperature three-quarters that of the screen in excess of the temperature of the walls, whereas region B will approximate to a mean excess temperature one-quarter that of the screen. These facts are easily understood when we remember that, since the effective distances from screen to strip and strip to glass wall are both $2c\lambda$, the temperature excess of the strip must be half that of the screen.

In practice the screens used did not possess a sufficiently great conducting power to maintain a perfectly uniform temperature. To calculate the temperature distribution over their surfaces, making allowances for the relative conductivities of the screens, the strips and the surrounding gas, would be a difficult problem into whose solution it is unnecessary to go for the purposes of the present Paper. It is here quite sufficient to proceed to the other extreme limit, and to assume the screen a non-conductor, and to work out the temperature of the strip on the supposition that a quantity of heat, sufficient to raise the temperature of a portion of the screen remote from the strip 1°C., is transmitted directly across from screen to strip. Applying this assumption to the central portions of the strip we are led to suppose that the isotherms are somewhat as represented in Fig. 15. It will be noticed that there are some easily explicable differences between these isotherms and those of Fig. 14.

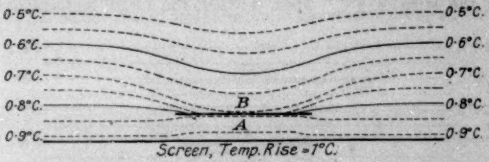

FIG. 14.—ISOTHERMS. GAS PRESSURE SUCH THAT $c\lambda$=DISTANCE STRIP TO SCREEN.

At the highest rarefactions the mean temperatures of regions A and B can be easily obtained if we remember that in effect, the distances from glass to screen, and from screen to strip are all $2c\lambda$. If, therefore, all the energy that passes from screen to strip passes on directly to the opposite glass wall, it follows that the temperature excess T of the strip must be half the temperature excess of the screen. Now, if the strip were absent, the amount of energy required to raise the screen 1°C. would be $k/c\lambda$, since heat is lost from both sides of the screen. Equating therefore $k/c\lambda$ to the quantity of heat transmitted from the screen to the near glass wall, plus the quantity transmitted from screen to strip, we have

$$\frac{k}{c\lambda}=\frac{2Tk}{2c\lambda}+\frac{Tk}{2c\lambda}$$

whence $T=\frac{2}{3}$.

The temperature excess of the screen is thus $1\frac{1}{3}$°C., whilst that of the strip is $\frac{2}{3}$°C. Thus, further, the mean temperature excess of region B above the temperature of the walls, will be 1°C., whilst that of region A will be $\frac{1}{3}$°C.

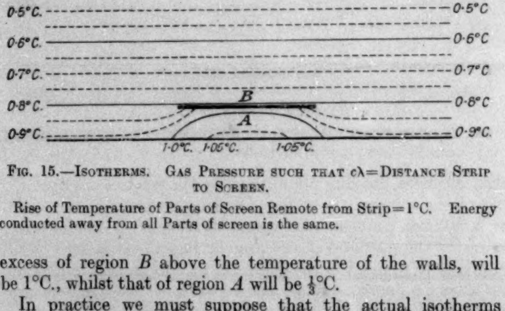

FIG. 15.—ISOTHERMS. GAS PRESSURE SUCH THAT $c\lambda$=DISTANCE STRIP TO SCREEN.

Rise of Temperature of Parts of Screen Remote from Strip=1°C. Energy conducted away from all Parts of screen is the same.

In practice we must suppose that the actual isotherms lie somewhere between those associated with the limiting cases described above. These isotherms change with reduction of pressure in a way quite different to those in Fig. 12, a noticeable difference being the fact that in the present case the mean temperature difference between the exterior and the interior increases progressively. Now, in the previous Paper, it was found that, at high pressures, a constant difference of temperature between various regions was associated with a thermal transpiration pressure varying rather more slowly than the inverse of the gas pressure. Hence, with an increasing temperature difference, as in the present instance, we should expect a more rapid variation. Such a variation is exhibited in the curves of Fig. 10.

It should be noted that, as a contrast to the previous case, the pressures on the two sides of the strip do not act in opposition. On the contrary, they co-operate, and we must, at high pressures, represent our systems of flow somewhat as in Fig. 16. It will be seen that the currents reduce the pressure in the region *B*, and increase it in the region *A*. At the very lowest pressures, moreover, the co-operation still exists, the mean temperature of *A* being above that of the exterior, and the mean temperature of *B* being below that of the exterior. If we make use of a result established in the previous Paper that when the mean free path of the molecules is very great, the pressures in various parts of an enclosure are proportional to the square roots of their absolute temperatures, we can easily calculate the resultant pressure on the strip. In fact, if ΔT be the rise of temperature of the screen, it can be easily shown that the pressure on the strip is given by $\Delta Tp/4T$, where T is the absolute temperature, and p the pressure of the gas in dynes. The dotted lines in Fig. 9 give values calculated from this formula for the two limiting cases previously described, and these lines appear as not improbable asymptotic limits of the experimental curves.

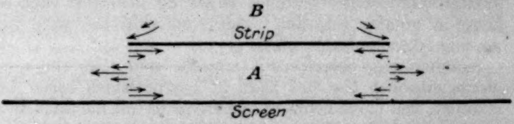

FIG. 16.—FLOW OF GAS ROUND STRIP PLACED IN FRONT OF HEATED SCREEN.

It may be remarked that the formula $\Delta Tp/4T$ was first obtained and verified by the Danish physicist Knudsen for pressures so low that the mean free path of the molecules was large compared to the distance of the strip to the heated walls. The present work, of course, deals chiefly with much higher pressures.

The theory hitherto employed postulates among other things a flow of gas from the exterior, along the surface of the strip. The assumption of such a flow is supported by the apparently satisfactory general explanations that can be given of the experimental results. More direct evidence can, however, be obtained. The gas can only flow as a result of an equal and opposite reaction on the material of the strip. In the case of a strip mounted parallel to the glass, as in Fig. 2, the two reactions on the opposite edges balance. If, however, we destroy this symmetry by mounting the strip at an angle to the glass, as in Fig. 4, there is the possibility that the unbalanced reaction on the strip might be of more importance than the repulsion from the glass, and thus cause the strip to move edgewise to the glass. This, of course, actually happens. When, however, the light is concentrated on the screen and not on the strip, as in Experiment VII., the repulsion effect is apparently too strong to permit such edgewise movement.

A further proof of the existence of the reaction on the strip is afforded by Experiment VIII. Here the reactions are in unstable equilibrium. If, as the result of a small mechanical shock, the symmetrical placing of the strip is disturbed, a greater temperature gradient is set up near one edge, and hence a greater reaction. This results in the strip again moving edgewise to the glass.

Summarising, therefore, it may be said that the experiments establish the fact that, at gas pressures below 1 cm. mercury, the movements of the strips of foil mounted in the various ways described, result neither from convection currents, nor from a difference of temperature of their surfaces, nor from " edge pressures." A previous Paper by the author on thermal transpiration, provides however an apparently satisfactory basis for the explanation of these results, and for others relating to variations of the deflections of the strips with the nature of the surrounding gas and with its pressure. Whilst at the lowest pressures conditions are comparatively simple, at the higher pressures they are complicated by gas currents of the thermal transpiration type. For the existence of such gas currents experimental evidence can be given.

The author has pleasure in thanking Prof. Lees for the facilities afforded for the prosecution of this research. The author has also to thank three of his students, Messrs. Andrews, Gregson and Harling, for the preparation of numerous curves and diagrams.

Abstract.

The present Paper arises out of two previous Papers by the author on the pressure of light (" Proc." Phys. Soc., XXV., p. 324, 1913, and XXVIII., p. 259, 1916), and consists of an experimental investigation of the nature of certain peculiar movements of strips of thin metal foil surrounded by rarefied gases, and exposed to radiation. The experiments deal

chiefly with phenomena at gas pressures below 1 cm. of mercury, and it is shown that the apparently diverse results obtained can be connected by a theory based on the work of a previous Paper (" Proc." Phys. Soc., XXXI., p. 278, 1919).

The author concludes that, at the highest rarefactions, the pressures on the strips arise from the fact that, if differences of temperature exist in an enclosure, the pressure of the gas is not uniform, but varies approximately as the square root of the latter's absolute temperature. The simple conditions that exist at low gas pressures are complicated, at the higher pressures, by gas currents which differ fundamentally from convection currents, but which are closely connected with the phenomena of thermal transpiration.

DISCUSSION.

Prof. **Eccles** said he was indebted to the author for his elucidation of gas action effects. He had frequently been puzzled by them in the past when designing certain instruments. In the curves shown the maximum effects occurred at a pressure of about $\frac{1}{100}$th cm. of mercury. What ratio did the mean free path of the gas at this pressure bear to the distance between the foil and the glass plate ? Had this ratio any particular bearing on the fact of the gas action being a maximum ?

Mr. **West** said the ratio was about one-sixth or one-fifth, though he could not be quite certain without reference.

Mr. F. E. **Smith** said that the author had experimented with two widths of foil. He suggested that experiments should be made with a series of strips of different widths and a curve obtained connecting the width of the strip and the deflection.

Mr. **West** said he had done this, but, except for broad strips, it was impossible to be certain that the temperatures were the same, as the disposition of the stream lines was different. It was necessary, therefore, to reduce the results to deflections per degree rise of temperature of the strip. At present his method of reduction was only rough, but the results showed that if the width of the strip is not comparable with the mean free path, the effect decreases as the width is diminished.

Mr. F. J. W. **Whipple** mentioned that when the author read his last Paper, Mr. Lewis Richardson also read one describing a new manometer. He gave a diagram connecting the force on the circular diaphragm of his instrument with the pressure, and afterwards he (Mr. Whipple) had compared Mr. West's transpiration theory with Richardson's diagram. According to the theory, the annular channel in the manometer was equivalent to a tube of 7 cm. diameter, which agreed very closely with the actual dimensions of the annulus. This appeared to afford confirmation both of Mr. West's theory and of Mr. Richardson's measurements.

The **Author**, in reply to Prof. Eccles, communicated that the maxima in the reduced curves corresponded to a mean free path somewhat greater than one-quarter the distance of the strip to the glass wall. An indication of the relations of the various maxima to the mean free path was given by the theory of the previous Paper. In reply to Mr. Smith, the author stated that experiments with narrower strips had been tried. Unlike the broad strips, however, the rise of temperature could not be assumed independent of the width, as the disposition of the stream lines was different. It was necessary, therefore, to reduce the results to deflections per degree rise of temperature of the strip. At present the method of reduction was only rough ; but, as far as they went, the results indicated that the effect decreased as the width was diminished. The previous results were thus confirmed.

XIX. *Absorption of Gases in the Electric Discharge Tube.* By F. H. Newman, *A.R.C.Sc., B.Sc., University College, Exeter.*

Received January 8, 1920.

1. *Introduction.*

The following experiments dealing with the absorption of gases in the discharge tube were made in continuation of those described in "Proceedings" Royal Society A., Vol. XC., 1914. It was shown that both hydrogen and nitrogen are absorbed in the discharge tube when sodium—potassium alloy is used as the electrode—whether cathode or anode. Also the amount of gas absorbed compared with the quantity of electricity passing in the secondary circuit was determined. It was found that the absorption increased as the pressure decreased, and was greater when the alloy was made the cathode than when it was the anode.

The alloy has properties similar to those of potassium and sodium, and so one would expect that these metals when used as electrodes in the discharge tube would also absorb nitrogen. As regards hydrogen, absorption cannot take place, as with this gas a scum appears at the surface of the alloy. By shaking the tube this scum can be removed, but in the case of the solid metals this cannot be done. The experiments now described were made to see if the metals absorb nitrogen, and if so how much gas is absorbed for different quantities of electricity passing in the secondary circuit—the measurements being made with various pressures of the gas. In order to measure the electricity, a small voltameter was employed, and the quantity of hydrogen gas liberated in it noted during a reading. The secondary discharge was made unidirectional by employing a cathodic valve in the secondary circuit of an induction coil.

2. *Preliminary Experiments.*

Fig. 1 shows the apparatus used and was similar to that employed in the previous experiments. The nitrogen was prepared in A by allowing air to stand over phosphorus for several hours. The gas then passed over phosphorus pentoxide in B for drying, and could be admitted into C—which was a known volume enclosed between two taps T_1 and T_2—by opening T_1. D was an oil manometer and E the discharge tube—with platinum wire electrodes—connected to a Toepler pump for exhaustion. In the experiments the apparatus was first exhausted, and the manometer calibrated as follows: Opening T_1 and then closing it a known volume of gas practically at atmospheric pressure was enclosed between T_1 and T_2. Opening T_2, the gas filled the apparatus and the alteration in pressure which it produced could be read from the manometer. By allowing several volumes to enter in this way, and noting the pressure before and after, the amount of gas which caused a recorded difference of pressure was found. So that when absorption took place, from the initial and final readings of the manometer the amount of gas at atmospheric pressure which had been absorbed was obtained. First sodium was tried. It was placed in a side tube *F* and on heating the latter the metal melted and ran down to the bottom of *E*, covering the platinum wire electrode. It was found, however, that no absorption took place under these conditions, owing to the impurity of the metal. To overcome this difficulty the discharge tube was silvered inside and sodium in a side tube *G* was distilled into *E* in a vacuum. By this means over the surface of the silver a layer of pure sodium was obtained. The same method was employed later with potassium.

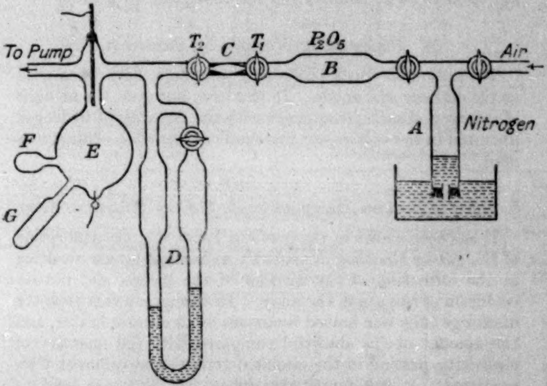

Fig. 1.

3. *Absorption of Nitrogen by Sodium.*

Observations were made with sodium as the cathode and later as the anode in the discharge tube. In both cases there was absorption of nitrogen, but the amount absorbed was greater when the metal was the cathode. The absorption increased as the pressure decreased, and although the amount of gas absorbed was many times greater than that absorbed by the walls of the vessel during discharge, yet it was not as great as when the sodium-potassium alloy was used. There appeared to be no falling off of the absorption with time.

4. *Absorption of Nitrogen by Potassium.*

Absorption also took place with potassium, both when used as the cathode and anode. In this case, however, the amount of nitrogen absorbed compared with the quantity of hydrogen liberated in the voltameter was smaller than when sodium was used.

5. *Effect of Heat on Absorption by the Sodium-Potassium Alloy.*

It has been shown in the previous Paper that the absorption of the gas by the alloy is probably a chemical action resulting in the formation of the nitrides of the metals and not an occlusion of the gas in the alloy. To strengthen this view the discharge tube was heated by means of an electric heater, and the amount of gas absorbed compared with the quantity of electricity passing in the circuit determined at different temperatures. It was found that when the alloy was used as cathode or anode, there was no difference in the rate of absorption from 50°C. to 200°C., but above this temperature the rate of absorption increased. Also when the discharge was stopped and the alloy heated to 300°C., no liberation of the nitrogen took place, so that the nitrides formed must be fairly stable.

6. *Tabulation of Results.*

A few of the results obtained are shown in the following tables, and for reference the amount of nitrogen absorbed in the case of the sodium-potassium alloy has been shown. The current in the secondary varied from 5 to 10 milliamperes, and the results were independent of the shape and size of the discharge tube.

TABLE I.—*Absorption of Nitrogen by Sodium.*

Sodium used as cathode.					Sodium used as anode.				
Volume of N_2 absorbed (A)	Pressure	Volume of H_2 liberated in voltameter. (B)	Value A/B.	Value A/B at same pressure for $Na-K$ alloy.	Volume of N_2 absorbed (A)	Pressure	Volume of H_2 liberated in voltameter. (B)	Value A/B.	Value A/B at same pressure for $Na-K$ alloy.
c.c.	mm.Hg.	c.c.			c.c.	mm.Hg.	c.c.		
0·36	8·1	0·52	0·7	2·3	0·27	6·5	1·63	0·2	0·7
0·18	6·6	0·21	0·9	2·3	0·54	5·1	2·70	0·2	0·9
0·54	5·5	0·51	1·1	2·4	0·19	3·7	0·53	0·4	1·4
0·35	3·4	0·30	1·2	2·6	0·37	2·2	0·72	0·5	1·9
0·55	1·0	0·42	1·3	2·7	0·36	0·7	0·60	0·6	2·5

TABLE II.—*Absorption of Nitrogen by Potassium.*

Potassium used as cathode.					Potassium used as anode.				
Volume of N_2 absorbed (A)	Pressure	Volume of H_2 liberated in voltameter. (B	Value A/B.	Value A/B at same pressure for $Na-K$ alloy.	Volume of N_2 absorbed (A)	Pressure	Volume of H_2 liberated in voltameter. (B)	Value A/B.	Value A/B at same pressure for $Na-K$ alloy.
c.c.	mm.Hg.	c.c.			c.c.	mm.Hg.	c.c.		
0·22	7·3	1·01	0·2	2·3	0·17	6·9	1·72	0·1	0·7
0·41	6·1	1·52	0·3	2·3	0·19	6·0	1·93	0·1	0·8
0·30	5·1	1·08	0·3	2·4	0·15	5·2	1·51	0·1	0·9
0·15	4·1	0·58	0·3	2·4	0·20	4·3	2·01	0·1	1·0
0·32	2·3	0·64	0·5	2·5	0·21	2·0	1·04	0·2	1·9
0·21	1·4	0·26	0·8	2·6	0·18	1·1	0·45	0·4	2·1

TABLE III.—*Absorption of Nitrogen by the Sodium-Potassium Alloy when used as the cathode at different temperatures and a pressure of* 4·0 *mm. Hg.*

Volume of N_2 absorbed. (A)	Volume of H_2 liberated in voltameter. (B)	Value A/B	Temperature.
c.c.	c.c.		
0·94	0·41	2·3	50°C.
0·67	0·29	2·3	90°C.
1·08	0·47	2·3	131°C.
0·77	0·34	2·3	200°C.
0·86	0·32	2·7	315°C.
1·29	0·26	4·9	360°C.

7. *Discussion of Results.*

Sodium and potassium absorb nitrogen in the discharge tube and this strengthens the view that the alloy produced by these two metals is a true alloy exhibiting properties similar to its constituents. The fact that it is liquid at ordinary temperatures being a case where the alloy has a lower melting point than either of its constituents. The liquid condition, however, probably accounts for the greater absorption of nitrogen for neither with sodium nor potassium is the absorption as great as with the alloy. As the temperature of the discharge increases the greater absorption in the case of the alloy is due probably to chemical combination, although it is an interesting fact that even at 300°C. no absorption of the gas took place with heat alone, so that the nitrides produced are stable. In all cases of absorption the surface of the metal must be very clean. With amalgams of sodium and potassium there is no absorption, the discharge being carried by the mercury vapour.

ABSTRACT.

The Paper describes a method of showing that absorption of nitrogen takes place in the discharge tube when sodium and potassium are used as the electrodes. The amount of gas absorbed for different quantities of electricity passing in the circuit is measured and is less than that absorbed by the sodium-potassium alloy. It shows also that when the alloy is used the amount of gas absorbed increases with temperature.

DISCUSSION.

Dr. **Beattie** referred to the bearing of the phenomena on photo-electric cells. He asked if it was possible to get such complete absorption as to give an X-ray vacuum, or if there was a dissociation pressure of the gas in equilibrium with the metal. The fact that the alloy absorbed more than either metal separately was possibly a surface tension phenomenon. If a compound was formed of which the surface tension was greater than that of the alloy, it would be drawn inwards, leaving a fresh surface of the uncontaminated alloy in contact with the gas. He had found that when a discharge passed in vacuum between iron electrodes a good deal of hydrogen was emitted. If the discharge was stopped this was re-absorbed, although had the discharge been continued the emission would have gone on as before.

Prof. **Fortescue** said the phenomena was probably the result of several different effects. For instance, if the volatilised films had not been at atmospheric pressure after volatilisation, the fact that they absorbed while the non-volatilised metals did not might be accounted for by the latter having retained gas previously absorbed while exposed to the atmosphere, while the former presented a fresh uncontaminated surface. It was further possible that the thickness of the sodium and potassium layers affected the results. As regards the anode and cathode effects, will this not depend on the relation between the numbers of positive and negative ions present? The author says there is no time effect. Does he mean by this that the absorption will go on, say, for 50 or 60 hours at a constant rate?

Mr. F. J. **Harlow** pointed out that the films of sodium and potassium were redistilled on silver. Might there not be alloys formed under these circumstances, the alloys and not the metals themselves, possessing the absorptive powers? Had the author used a lime cathode? With this it was possible to get a very steady discharge from the lighting voltage.

Mr. B. P. **Dudding** thought we were here dealing with problems which Langmuir and others had worked out at some length. Some recent work of Dr. Norman Campbell showed that, with suitable discharge conditions, compounds of almost any two elements could be formed. The essential thing was to remove the products of the reaction.

Mr. **Newman**, in reply, said that it was not possible to get a very high vacuum with hydrogen. Although it was absorbed by the alloy it was impossible to get rid of the last traces. The surface of the alloy must be very clean. When the discharge is in nitrogen, the surface keeps clean, the scum formed breaking up and going to the sides of the tube, but with hydrogen a black scum forms which requires to be mechanically removed by shaking the tube. As regards the constancy of the rate of absorption, he could not give actual figures, but the rate certainly did not diminish for a very long time. He agreed that it would be difficult to prove that the sodium and potassium films had not alloyed with the silver. He had not tried a lime cathode, but had got very steady discharges by using the type of Wehnelt interrupter, which he had recently shown to the Society.

XX. *A Directional Hot-Wire Anemometer of High Sensitivity, especially Applicable to the Investigation of Low Rates of Flow of Gases.* *By* J. S. G. Thomas, *M.Sc.* (*Lond.*), *B.Sc.* (*Wales*), *A.R.C.S.*, *A.I.C.*

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(Communicated by D. Owen, B.A., D.Sc.)

A type of directional hot-wire anemometer was introduced by the author* for the purpose of readily determining the direction of flow of air or other fluid in pipes or other channels. Essentially the instrument consists of two fine platinum wires arranged parallel and one behind the other in close juxtaposition, transversely to the direction of flow of the gas in the pipe or channel. The wires constitute two arms of a Wheatstone bridge, the remaining arms being formed of a resistance of 1,000 ohms and an arm capable of adjustment. Throughout, a constant current is maintained in the bridge, and the battery terminals are connected to the bridge at the respective ends of the platinum wires, so that the maximum heat is developed in these wires. A galvanometer is inserted in the bridge in the usual manner. A current from 1·0 to 1·5 amperes is suitable for use with the bridge, the wires being thereby heated to a temperature of about 300°C. The operation of the present type of hot-wire anemometer as an indicator of the direction of flow of fluid in the channel wherein it is inserted is dependent upon the fact that of the two fine platinum wires freely exposed to the cooling effect of the current of fluid, that one experiences the greater cooling effect upon which the stream of gas is first incident, this wire exercising by its presence a shielding effect upon the second wire, so that the latter is less cooled by the stream of fluid. A current of fluid being established in the channel, the galvanometer deflection is reversed on reversing the direction of flow of the fluid in the channel. Such a hot-wire anemometer, therefore, affords a ready means of ascertaining the direction of flow of fluids in various units of a complicated network of gas or other mains. Subsequent experience with anemometers of this type has shown that they possess special characteristics which make then particularly useful in the investigation of very low rates of flow. In the region of low velocities, serious difficulties are encountered in the use of hot-wire anemometers of the types introduced either by King[*] or Morris,[†] owing to the free convection current of fluid ascending from the heated platinum wire. This point it is not proposed to deal with here; it is treated in detail in the course of an investigation which will appear shortly. Here it is sufficient to state that the disturbing effect of such free convection current on the indications of the instrument is considerably reduced by the use of a second wire shielded from the cooling effect of the stream of fluid, as introduced by Morris to afford temperature compensation in the bridge arms. The effect of the free convection current upon the instrument's indications is, of course, the more completely eliminated, the more nearly the state of the protected wire constituting the compensating arm resembles what may be termed that of the exposed arm of the bridge. The type of directional anemometer described above approaches more nearly this ideal condition of affairs than is the case with Morris's type of instrument. Complete temperature compensation is afforded by the close juxta-position of the wires—in fact, the stability of the zero of the instrument even when used with currents of different magnitudes is a marked characteristic of the instrument. This point is likewise discussed in detail in the work referred to above. The purpose of the present communication is to draw attention to the fact that contrary to expectation, the sensitiveness of the type of directional anemometer described, when subjected to the cooling effect of a slow-moving stream of fluid, is much greater than that employing one freely exposed wire and a second entirely shielded from the cooling effect of the stream of fluid, as embodied in the type introduced by Morris.

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* Journal Soc. of Chem. Ind., 1918.

Experimental.

For the purpose of the present experiments, an anemometer tube was prepared so that the same wire could be used either as the first exposed wire of a pair constituting a directional anemometer of the type described, or as the exposed wire of a pair, the other member of which was shielded by insertion in a shielding tube. The device is shown in Fig. 1. The directional anemometer is constituted of the fine platinum wires *a* and *b*. The other type of anemometer is constituted of the wires *a* and *c*. The wires *a*, *b* and *c* were inserted at such distances from one another that no appreciable disturbance in the flow was experienced at *a* owing to the presence of *b* and *c*. The mode of insertion of the wires in the tube will be evident from the diagram. Efficient insulation of the wires was afforded by the use of ebonite plugs as shown. *c* was secured in position by insertion through holes in the copper rods *D* and *E*. The wires *a* and *b* were secured in slightly different manner as shown, necessitating the provision of strengthening pieces of copper rod driven into the ebonite plugs. The ends of all plugs were carefully shaped, so that no discontinuity in the surface of the tube was produced by their presence. The wires *a*, *b* and *c* were all cut from the same sample of platinum wire,* and were as nearly as possible of the same length, equal to the diameter of the tube, so that they were as nearly as possible of equal resistance. They were attached to the copper rods by means of the minimum amount of silver solder affording a secure junction, and were artificially aged by the passage of a current of 1·5 amperes for two hours. The ends of the tube were connected to 5 ft. lengths of similar tube by means of spigot unions of special design affording a smooth junction. The bridge connections were so made that either *b* or *c* could be inserted in the bridge as desired. A voltmeter of resistance 1,000 ohms could be inserted in parallel with *a*, whereby the resistance of *a* could be calculated from a knowledge of the current of the bridge, this being maintained constant throughout by means of a rheostat inserted in the battery circuit.

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[*] "Phil. Trans.," 1914, A. 520, 373-432.

[†] B.A. Report, September, 1912.

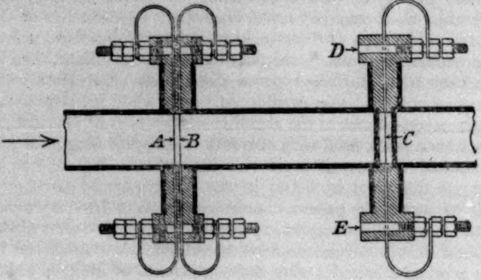

FIG. 1.

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* The platinum wire was of purity 99·5 per cent. A supply of 100 per cent. pure platinum could not be obtained owing to war conditions.

TABLE I.—*Result Employing Shielded Wire Non-directional Anemometer. Current* 1·1 *Ampere.*

Temperature.		Barometer (in.).	Vol. of air passed (1/60 cub. ft.).	Time of obser. (min.)	Uncor. volume per hour (cub. ft.).	Deflection.			Balance resistance (ohms).		Drop of potential across first wire (volt.).		Cor. vol. of air per hour (cub. ft.) at 0° and 760 mm.	Equiv. mean velocity in pipe (cm. per sec.)	Resistance of wire (ohm).			
															Zero flow.		With flow.	
Air stream (°C.).	Meter (°F.).					Reading to left.	Reading to right.	Diff.	Zero flow.	With flow.	Zero flow.	With flow.			First wire.	Second wire.	First wire.	Second wire.
19·0	63·5	29·44	1·68	1	1·68	228	273	45	990	982	0·748	0·741	1·52	2·61	0·680	0·687	0·674	0·687
19·4	64·0	29·44	4·62	2	2·31	202	285	83	989	972	0·750	0·734	2·09	4·96	0·682	0·690	0·667	0·688
19·3	64·0	29·47	5·01	2	2·50	175	322	147	992	968	0·750	0·729	2·27	5·39	0·682	0·687	0·663	0·685
19·4	64·5	29·47	5·29	2	2·65	161	344	183	991	958	0·747	0·720	2·40	5·70	0·680	0·686	0·655	0·684
19·3	64·8	29·50	5·52	2	2·76	132	370	238	995	952	0·748	0·715	2·50	5·93	0·680	0·683	0·650	0·683
19·8	64·8	29·50	6·21	2	3·10	110	403	293	994	939	0·750	0·705	2·81	6·67	0·682	0·686	0·641	0·683
20·0	65·0	29·51	6·40	2	3·20	71	427	356	995	935	0·750	0·700	2·90	6·88	0·682	0·685	0·636	0·680
20·1	65·4	29·51	3·75	1	3·75	36	480	444	995	917	0·750	0·689	3·40	8·07	0·682	0·685	0·626	0·683
20·1	65·1	29·54	3·88	1	3·88	19	502	483	994	913	0·750	0·680	3·52	8·36	0·682	0·685	0·618	0·677
								Mean	0·993	Mean	0·749			Mean	0·681	0·685	Mean	0·683

TABLE II.—*Result Employing Double Exposed Wire Directional Anemometer. Current* 1·1 *Ampere.*

Temperature.		Barometer (in.).	Vol. of air passed (1/60 cub. ft.).	Time of obser. (min.)	Uncor. volume per hour (cub. ft.).	Deflection.			Balance resistance (ohms).		Drop of potential across first wire (volt.).		Cor. vol. of air per hour at 0° and 760 mm. (cub. ft.)	Equiv. mean velocity in pipe (cm. per sec.)	Resistance of wire (ohm).			
															Zero flow.		With flow.	
Air stream (°C.).	Meter (°F.).					Reading to left.	Reading to right.	Diff.	Zero flow.	With flow.	Zero flow.	With flow.			First wire.	Second wire.	First wire.	Second wire.
18·9	63·5	29·44	1·12	4	0·28	231	271	40	996	988	0·754	0·750	0·25	0·59	0·685	0·688	0·682	0·690
19·4	63·8	29·44	1·66	4	0·42	210	291	81	997	982	0·754	0·750	0·38	0·90	0·685	0·688	0·682	0·695
19·5	64·0	29·47	1·72	3	0·57	192	310	118	996	975	0·754	0·749	0·52	1·13	0·685	0·688	0·681	0·698
19·4	64·0	29·47	2·20	3	0·73	180	332	152	995	966	0·754	0·750	0·66	1·57	0·685	0·688	0·682	0·706
19·7	64·8	29·50	1·88	2	0·94	146	355	209	995	957	0·752	0·750	0·85	2·02	0·683	0·686	0·682	0·713
19·8	64·8	29·50	2·54	2	1·27	105	397	292	996	944	0·753	0·740	1·15	2·73	0·684	0·687	0·673	0·713
19·9	65·0	29·51	2·83	2	1·42	92	420	328	995	939	0·752	0·741	1·29	3·06	0·683	0·686	0·673	0·717
20·1	65·2	29·51	3·66	2	1·83	40	464	424	997	924	0·754	0·730	1·66	3·94	0·684	0·687	0·664	0·719
20·0	65·2	29·54	3·82	2	1·91	30	470	440	998	922	0·754	0·730	1·73	4·11	0·684	0·687	0·664	0·720
19·5	65·0	29·54	2·43	1	2·43	56	450	394	998	927	0·754	0·711	2·20	5·23	0·684	0·687	0·646	0·697
19·5	65·0	29·54	2·57	1	2·57	98	408	310	997	940	0·754	0·712	2·33	5·53	0·684	0·687	0·647	0·688
19·5	65·0	29·54	5·70	1	5·70	170	333	163	997	963	0·754	0·650	5·17	12·28	0·684	0·687	0·591	0·614
18·8	65·1	29·54	9·35	1	9·35	207	295	88	997	977	0·754	0·592	8·48	20·03	0·684	0·687	0·538	0·551
18·8	65·0	29·55	11·30	1	11·30	217	284	67	997	981	0·754	0·580	10·25	24·32	0·684	0·887	0·527	0·537
18·8	65·0	29·55	20·50	1	20·50	230	271	41	997	981	0·754	0·530	18·60	44·14	0·684	0·687	0·482	0·491
								Mean	997	Mean	0·754			Mean	0·684	0·687		

[*To face* p. 199.

The third arm was adjusted throughout to 1,000 ohms. The bridge being balanced with zero flow of air, the voltmeter was inserted, and drop of potential across *a* measured. A current of dry air was established in the tube. This was derived from a weighted gas holder of 5 cubic ft. capacity, provided with automatic pressure compensation. The air was dried by passage through a column of calcium chloride, its temperature read, and the volume passing down the tube determined by a wet gas meter, by Sugg, of 1/12 cubic ft. capacity, this being standardised by means of the 1/12 cubic ft. bottle prescribed by the Metropolitan Gas Referees. The rate of delivery of the air was extremely steady, the galvanometer deflection not varying by more than one division for any definite rate of flow. Confirmatory readings of the flow of air were taken in all cases. Care was taken that no leakage occurred at the various joints of the apparatus. The drop of potential across *a* was again measured, and the balancing arm adjusted so that the bridge was again balanced. *a* and *c* were now inserted in the bridge, and the readings repeated. A series of similar observations was made for various mean velocities of flow of dry air in the tube. All volumes are reduced to 0°C. and 760 mm. pressure (dry).

Results.

Diameter of tube	2·0534 cm.
Diameter of wire employed	0·0040 in.
Distance between *a* and *b*	0·1 cm.
Distance between *a* and *c*	7·6 cm.

The respective volumes (at 0°C. and 760 mm.) are converted to the equivalent mean velocities of the air stream (reckoned at 0°C. and 760 mm.) by multiplying the former as given in cubic feet per hour by 2·374. The ratio arm in the bridge was maintained throughout equal to 1,000 ohms, and the sensitiveness of the galvanometer was reduced by shunting with 10 ohms, so that a full scale deflection was obtained as the maximum deflection using the directional anemometer. The current employed was 1·1 ampere. The results obtained and calculations based thereon are set out in Tables I. and II., and are represented graphically in Figs. 2 and 3.

Discussion of Results.

From Fig. 2 it is seen that the deflection obtained when using the directional type of anemometer is, except for the extremely low values, strictly proportional to the mean velocity of flow until a maximum deflection is reached at *b*, corresponding to a mean velocity of very approximately 4 cm. per second. Thereafter the deflection diminishes with increasing velocity. It is seen from the form of this portion of the curve that the deflection does not vanish for very high values of the mean velocity of flow in the pipe. The curve *OCD* is the curve obtained by employing the type of non-directional anemometer, the leading wire being the same as in the directional type. The greater sensitiveness of the directional type is clearly indicated. Thus, with the directional type of anemometer, a mean velocity of about 4 cm. per second produces a deflection of 430 scale divisions. The deflection under similar circumstances using the non-directional type is 62. Over the range from *A* to *B* the directional type is seen to possess a sensitiveness approximately seven times that of the non-directional type of anemometer. The author proposes to discuss in detail in the Paper referred to above the form

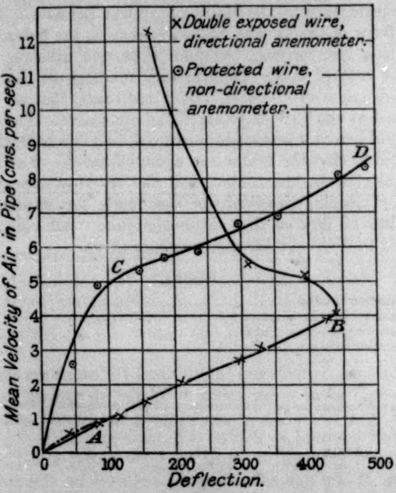

Fig. 2.

of the curve *OCD* obtained with the non-directional type of instrument. The form near the origin is determined principally by the magnitude of the free convection current arising from the wire. The almost exact parallelism between the portions *AB* proceeding from the origin, and *CD* of the respective curves indicates that, except in the immediate neighbourhood of the origin, the effect of the free convection current is almost completely compensated for by the use of the directional type of instrument. The argument can be developed mathematically, thus: If V_s is the velocity of the stream, and V_c the velocity of the free convection current arising from the wire, the effective velocity of the cooling current of air is $\sqrt{V_s^2+V_c^2}$, with the wire disposed horizontally as in the present experiments. In the case of the wire enclosed in the protecting device, the cooling effect on the wire from its state in an absolutely stagnant medium is entirely due to the free convection current, which we may assume, at least for small values of the impressed velocities of the air stream to be equal to V_c. The differential cooling effect experienced by the wires is therefore that due to a velocity $\sqrt{V_s^2+V_c^2}-V_c$. The author shows in the Paper referred to that in the present case, V_c is of the order 15 cm./sec. Hence, for small values of V_s the differential cooling effect experienced by the two wires is that due to an effective velocity on one wire equal to $V_s^2/2V$, and is thus seen to be determined by both V_s and V_c. This latter is itself dependent on V_s as the temperature of the wire is determined partly thereby. For large values of the impressed velocity of the stream, the differential cooling effect is equal to that due to an impressed velocity $V_s+\frac{V_c^2}{2V_s}-V_c$ and for large values of V_s this is seen to be practically independent of V_c. With the directional type of anemometer, on the other hand, both wires are exposed to the effect of the air stream and their respective free convection currents. The results can best be discussed in terms of the curves shown in Fig. 3, in which the resistances of the respective wires are plotted as ordinates against the impressed velocities of the air stream as abscissæ. Considering first the broken-lined curves *C* and *D* representing the variation of the resistance of the leading wire and the protected wire in the non-directional type of anemometer respectively, it is seen that the resistance of the protected wire, and consequently its temperature is, within the limits of experimental error, constant. The resistance and consequently the temperature of the leading wire falls continuously as the impressed velocity of the air stream is increased. Considering now the curves *A* and *B*, which represent respectively the variation of the resistance of the leading and second wires (*a* and *b*, Fig. 1) of the directional type of instrument, it is seen that the resistance of the leading wire

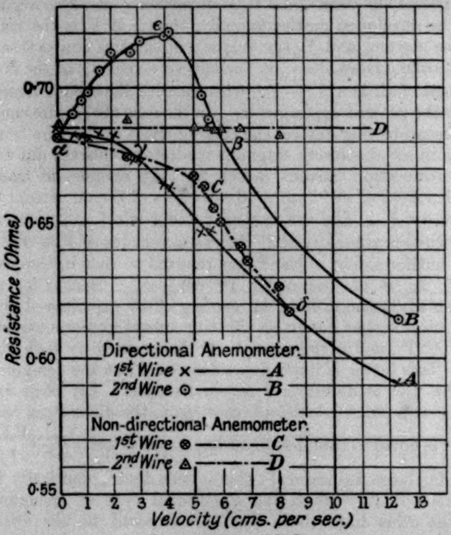

FIG. 3.

(*a*, Fig. 1) falls continuously with increasing impressed velocity of the air stream, whereas that of the second wire (*b* Fig. 1) first increases, reaches a maximum at ε, and thereafter diminishes continuously. The greater vertical distances in their initial portions between the curves *A* and *B* referring to the wires of the directional anemometer, compared with the distance between *C* and *D* in the same region affords a ready explanation of the greater sensitiveness in the region of low impressed velocities, of the directional type of instrument. The curves *A* and *B* have two outstanding features when compared respectively with the curves *C* and *D*. These may be characterised thus: The curve *C* shows that over the region of velocities embraced within $\gamma\delta$, the leading wire *a* is more cooled when employed in the directional instrument than when used as a component of the non-directional type of instrument. The curve *B* shows that, contrary to expectations, the second wire *b*, over the region of impressed velocities, represented by the curve from α to β, far from having its temperature reduced by the stream of air, is considerably warmer when the stream passes over it than when the air is not flowing. These facts are explained as follows: The air reaching *b* in the stream has already passed over *a*, and been thereby heated. The radiation loss from *b* is therefore less when the stream flows than when the medium surrounding *b* is still. This diminished radiation loss entails a rise in the temperature of *b*, which is accompanied by an increased loss by free convection from *b*. The final temperature of *b* is determined jointly by these two causes. Initially with low impressed velocities, the air carried from *a* to *b* is at a comparatively high temperature, and the radiation loss from *b* is considerably diminished. The rise of temperature of *b* due thereto is not materially diminished by the accompanying increased loss due to free convection, and the temperature of *b*, on balance, rises. With increasing impressed velocities, the temperature of the air carried from *a* to *b* increases owing to the fact that the time of transit of the air from *a* to *b* is diminished with consequent smaller drop of temperature in the air during the transit. Another factor operative in the reverse direction as regards the radiation from *b* requires attention. With increasing impressed velocity, the actual temperature of the air leaving *a* diminishes continually. The actual temperature of the air in the stream in the neighbourhood of *b* is therefore determined by (1) the velocity of the stream and (2) by the time of transit which determines the loss during transit. Initially with low velocities, the actual temperature of the air in transit is such that on balance, the effect of the high temperature due to the small motion is increasingly predominant over the subsequent drop in transit from *a* to *b*. A point is therefore attained with increasing impressed velocity such that the temperature of *b* attains its maximum value. Thereafter, the impressed velocity is such that the counter effects of initial heating of the air at *a* and subsequent cooling during transit from *a* to *b* operate so as to reduce the temperature of *b* owing to the growing predominance of the latter effect. When such an impressed velocity is attained that the heat acquired by passage over *a* is subsequently exactly that lost during the time of transit from *a* to *b*, it is clear that the temperature of *b* is exactly what it would be in the absence of any impressed velocity. This condition is represented by the point β of the curve B. Thereafter with increasing impressed velocity of the stream, the temperature of b is continuously reduced by the stream, as shown by the portion of the curve βB. Considering now the portion of the curves A and C, it is seen that the temperature of the wire *a* over this region of impressed velocities is actually *lower* when the adjacent wire *b* is heated than when *b* is not heated. This phenomenon is explained as follows: When *a* and *b* are both heated, and an impressed stream of air passed over them, the cooling effect experienced by them is the resultant effect of the impressed velocity and the free convection currents arising from the wire. It is readily seen that the resultant current of air from the wire *b*, which originally rises vertically upwards, therefrom when the impressed velocity of the air stream is zero, tends towards the horizontal as the impressed velocity is increased. This free convection current from *b* produces a cooling convection current in the neighbourhood of *a*, so that when *b* is heated by the electric current, the cooling due to this induced convection current in the neighbourhood of *a* more than counterbalances the heating of *a* due to radiation from the heated wire *b*. The temperature of *a* is, under these circumstances, actually reduced below what it would be if *b* were not heated.

It should now be clear that the extreme sensitiveness of the directional type of anemometer, at low velocities, arises partly from the increased temperature of the second wire due to causes originating in the first wire of the pair, and partly to the diminished temperature of the first wire originating in the second wire.

The directional type of anemometer, with its almost perfect compensation of free convection effects is of especial application in the investigation of low velocities of flow. A still more sensitive arrangement than that described is afforded by utilising two couples of wires in close juxta-position, the couples being separated by an interval and constituting therefrom a Wheatstone bridge. Thereby the sensitiveness is increased a further two-fold.

The work detailed herein was carried out in the physical laboratory of the South Metropolitan Gas Company, and the author desires to express his thanks to Dr. Charles Carpenter for the ready provision of the necessary facilities, and for permission to publish the results obtained.

ABSTRACT.

The instrument consists of two fine platinum wires mounted close together, and forming two of the arms of a Wheatstone bridge. These are heated by the current in the bridge. When a stream of gas moves in a direction perpendicular to the wires but parallel to the plane containing them, the leading wire is cooled, while the second wire, being shielded by the first, is not cooled as much, and may actually be heated on account of the air flowing past it being warmed by the first wire. A deflection of the galvanometer is obtained, therefore, which is reversed if the flow of gas is in the reverse direction. For low rates of flow of gas, the instrument is much more sensitive than the non-directional hot-wire anemometer. An explanation of the increased sensitivity is advanced.

DISCUSSION.

Dr. **Ezer Griffiths** said he had been working with hot-wire anemometers recently, but they required somewhat complicated gear. It is assumed in working out the theory that the flow of gas is parallel to the axis of the tube. What happens if this is not so owing to the proximity of bends?

Dr. **Tucker** said he had used a hot-wire anemometer for impulsive and alternating flows. But the hot-wire inevitably produces its own direct current due to convection. He had found the curious result that the sensitivity for alternating currents could be increased many times by increasing the strength of the direct current to which the wire was exposed. Had the author attempted to increase the sensitivity by reducing the diameter of the wires? The sensitivity rose very rapidly as the diameter was reduced.

Dr. E. H. **Rayner** asked if experiments had been made with different dimensions to see if the range of usefulness could be increased. He thought the velocities in gas mains would usually be much greater than 4 cm. per second, which seemed to be about the maximum useful range with the present instrument.

Mr. F. E. **Smith** said he was not convinced by the author's explanation of the first wire being further cooled, due to the proximity of the second hotter wire. He suggested a simple experiment to prove whether or not this cooling actually took place.

Dr. **Borns** asked if all the experiments had been in coal gas.

Prof. **Rankine** asked if any difficulties arose due to the wires not remaining in the plane parallel to the flow. One would expect sagging to occur, and the two wires would not necessarily bulge in the same direction.

Dr. D. **Owen** suggested that there might be some advantage in this respect if grids were used instead of wires.

Mr. J. Guild asked as to the variation in the readings of the instrument due to alteration of the inclination of the plane of the wires to the stream of gas. To within what limits could the direction of flow be determined?

Dr. L. Hopwood said the author was to be congratulated in finding a directional instrument which was more sensitive than a non-directional one. Direct measurements could be made on the convection current effects by using a vertical flow. In one direction the convection currents would assist the flow, and in the other would be against it.

Dr. T. Barratt (communicated): I have read Mr. Thomas's interesting Paper with much pleasure. His explanations of the more or less intricate phenomena involved are convincing and simple. In one particular, however, the facts can be explained more simply as well as more accurately. Radiation from thin wires, even at moderately high temperatures, forms a very small percentage of the total heat lost. In the case of Mr. Thomas's anemometer wires, this percentage is probably less than 5, and certainly less than 10 if the wire is "bare"—*i.e.*, slightly soiled by exposure to air, &c. That is to say, nearly all the lost is lost by convection. An explanation of the observed facts can be made, however, without any assumption of radiation at all. Assuming that nearly all the loss of heat is due to convection, "free" or "forced," the air reaching *b* from the direction of *a* is considerably warmer than the rest of the gas. Hence the convection from *b* (which strictly obeys Newton's first power law) is considerably reduced, resulting in a higher temperature for the wire *b*. At higher velocities of the cooling fluid the air from *a* reaching *b* is not so warm, so that *b* is cooled more than at low velocities. The remaining explanations can follow on the same line of reasoning.

The Author, in reply to Dr. Griffiths, stated that as regards disturbed stream lines, King states ("Phil. Trans.," A.520, 1914, p. 407) that the experiments of A. M. Gray show that the hot-wire anemometer renders possible a consistent measure of turbulent flow where other methods of measurement would be inadequate. King ("Phil. Trans.," A.520, p. 425) has studied the effect of inclination of the hot wire to the stream on the convection constants of small wires. With the directional type of instrument, the maximum shielding of the second wire is obtained when the wires are normal to the gas stream. The point has been investigated by the author, and the variation of galvanometer deflection with variation of the plane of the wires to the free convection current arising from the wires has been made the basis of the construction of an electrical inclinometer. The variation of sensitivity with diameter of the wire as mentioned by Dr. Tucker in the case of the non-directional instrument has been carefully investigated by King ("Phil. Trans.," *loc. cit.*, p. 383 *et seq.*). In the case of the directional instrument, the increased sensitivity due to the use of a given wire is somewhat reduced by the smaller shielding effect they afford. In reply to Dr. Rayner, the directional type of instrument is not intended to be used as a measurer of gas flow, save in its special sphere of sensitiveness, viz., for low rates of flow of gas. Certain operations in gas works are carried out in the mains only when the gas flows in one direction. This direction being indicated, the actual velocity in the main can, if need be, be ascertained by a non-directional type of instrument. The author's experiments were not confined to coal gas, but embraced experiments using currents of air, carbon di-oxide, oxygen, hydrogen and other gases. Difficulties are encountered when using a bare wire anemometer with gases such as hydrogen or methane, &c., particularly if the wires are used at an elevated temperature. These are entirely eliminated if the wire is coated with glass. The instrument is then considerably more robust and its sensitivity little reduced. The experiment suggested by Mr. Smith had been tried and the cooling effect confirmed. The existence of the cooling effect seemed to depend upon the relative disposition of the wires as regards distance apart. The sagging referred to by Dr. Rankine did occur, but its influence could, with care, be eliminated. This was done by ensuring that the wires, initially in a diametral longitudinal section of the pipe, remained so when heated. The necessary adjustment is easily effected by rotating the rods to which the wires are affixed. In any case, the sagging can be reduced by employing a small heating current, and can be eliminated entirely by employing a spring-fork support for the wires as introduced by King. The author agreed with Dr. Owen that the use of a grid of parallel wires would reduce the variation due to sagging. Experiments on the lines suggested by Dr. Hopwood have already been carried out by the author, and direct measurements made of the magnitude of the velocity of the free convection current arising from fine heated platinum wires. The results obtained enable the temperature coefficient of the free convection velocity to be very accurately determined. Mr. Guild's query is answered in the reply to Dr. Griffiths. The author is unable to assent to the explanation of the phenomena suggested by Dr. Barratt. Dr. Barratt's experiment on the radiation and convection losses from a heated wire have extended only up to a temperature (100°C.) considerably below those employed in the present case. It is interesting to note that he finds ("Proc." Phys. Soc., 1915, 28, p. 10) that the proportion of radiation is greater at higher temperatures. In the absence of data, it can hardly be claimed that omission of all consideration of radiation, even though this amount to only 5 per cent. of the total heat loss involved, can afford an accurate interpretation of the results. In the first place, it is not the actual magnitudes of the respective losses by radiation and convection that are involved, but rather the variations of these with variation of the temperature difference between the wire and its surroundings under the peculiar conditions of temperature distribution around the wire in the present experiments. In this connection it must be noted that the *resultant* direction of flow of the hot stream of gas from the first wire is such that the distribution of temperature round the second wire due thereto is asymmetric, and with low rates of flow of gas the temperature difference between the medium above the wire and that below is increased on this account, the hot current of gas from the first wire being directed towards the region above the second wire, while the convection current in the immediate vicinity of the second wire is little disturbed, owing to the shielding influence of the first wire. There is, therefore, an increased convection current from below upwards across the second wire, with a consequent *increased* convection loss of heat, and not a *decreased* loss, as suggested by Dr. Barratt.

XXI. *Experiments with a new Micro-Balance.* By Dr. Hans Pettersson.

Received April 12, 1920.

(Communicated by A. W. Porter, F.R.S.)

Considering the fundamental importance of accurate weighings both for physical and chemical research it is surprising how little progress had been made until about ten years ago with regard to improving the sensibility of weighing balances. A certain number of micro-balances had been constructed by Angström, Salvioni, Nernst and others; but their absolute sensibility rarely exceeded a thousandth part of a milligram. Owing to the very delicate mode of suspension chosen for these instruments they were extremely fragile, so that the maximum load which they could safely carry was comparatively small. Their not very considerable gain in *absolute* sensibility was therefore compensated by a disproportionate loss in *relative* sensibility, which contributed to make their use very limited. The most widely known of these early micro-balances, the Nernst balance, could only be charged with a few cg., which it weighed to about 0·001 mg.

Quite a new departure in the technique of micro-weighings was made in 1911, when Steele and Grant published a description of their remarkable all-quartz micro-balance.* By making the beam, as well as the weights and counterpoise, entirely from fused, non-corrodible, and practically non-hygroscopic silica, they were able to give their instrument a very high constancy of its zero. The mode of balancing it on a minute knife-edge of ground quartz made it susceptible to extremely small changes in weight, which could be counterbalanced and measured with a sufficient degree of accuracy, thanks to an ingenious method of aerostatic compensation, namely by varying the pressure of the air inside the case, so as to change the buoyancy of a small sealed silica bulb of known capacity (the "air-weight") suspended from the balance.

Given a certain dexterity on the part of the operator, these instruments could be put to excellent use, as was amply proved by the brilliant work of Sir William Ramsay and Dr. Whytlaw-Gray on the density of niton and on the atomic weight of radium.†

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* Proc. Roy. Soc., A. 82, p. 580.

† Proc. Roy. Soc., A. 84, p. 536, and A. 86, p. 270.

In the course of these investigations a tiny gold capsule, after being heated red hot, was found to undergo a curious loss in weight, which it slowly regained on cooling. This effect was first supposed to be due to a liberation of air adsorbed by the metal; and when my work at University College, London, was begun in the autumn of 1911 a closer study of this phenomenon was the subject suggested to me by Sir William Ramsay. For this research the quartz micro-balance was to be used, and I was incidentally to try to improve it, so as to make it stronger and less difficult to work with, by suspending it by metal fibres instead of balancing it on a knife-edge.

The advantages of this arrangement appeared somewhat doubtful, considering the rather imperfect elastic properties of metals. Still the method had to be tried. The technical difficulties of soldering thin Wollaston wires to the platinized silica rods of the beam (an operation carried out under the microscope with the aid of a minute electrical soldering apparatus), proved to be unexpectedly great. Several months of vain attempts finally resulted in a complete quartz balance suspended by platinum fibres about 0·01 mm. thick. The suspension, however, was entirely unsatisfactory; even at a comparatively low absolute sensibility the zero was found to undergo considerable variations.

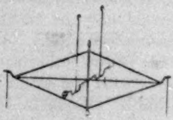

FIG. 1.

Before falling back on the old knife-edge balance I made an attempt to suspend a beam by fine *silica* fibres drawn out from the beam itself. The first balance of this type gave very good results, and my subsequent investigations, of which a short account is to be given in the following, have been carried out with similar instruments. The properties of the fibre suspension have been studied theoretically and by experiment. The technique of the instrument itself and of the weighings has been developed. Finally, a fibre-suspended micro-balance has been used in some physical investigations.

It is not possible for me to express adequately my feelings of gratitude to my late teacher, Sir William Ramsay, under whose inspiring influence I have been allowed to work for nearly one year. Probably only those who have themselves had the privilege to work under this " prince de la science " will be able to realise it.

*The Fibre Suspended Micro-Balance.**

After about ten months spent at University College these investigations were continued in the physical laboratory of Stockholms Högskola, its director, Prof. C. Benedicks, kindly putting the necessary instrumental resources at my disposal. With the aid of the skilful mechanic of that institution, Mr. C. Andersson, a new type of case was constructed. Fig. 2 represents a vertical section through a similar case, made for a balance of *reduced* size by the firm Lyth of Stockholm, who are to manufacture instruments of this kind.

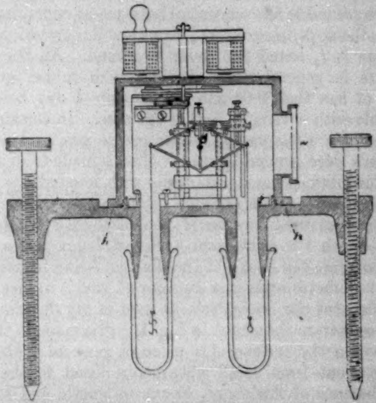

FIG. 2.—VERTICAL SECTION OF MICRO-BALANCE (TO HALF SCALE).

The balance-cases used by Ramsay & Gray had a solid brass rod, by which the arrestment was worked, ground into a conical jacket through the floor of the case and made air-tight with rubber-grease. On the new case the arrestment is worked without any material transmission, simply by rotating an "electro-magnetic key" on the top of the brass cover, so that a soft-iron anchor-piece is carried round inside by the magnetic forces. In this manner the beam can be lifted off its central fibres when not in use, or made to depend from them when a reading is to be taken.

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* *See* my thesis :—" A New Micro-balance and its use." Göteborg's "Vet. of Vitterh. Samhalle's Handlinger," XVIth series, 1914.

The supports which grip the silica hooks at the ends of the central fibres can be moved by screws in order to adjust the beam to its correct position, so that the end-fibres depend straight into the centre of the detachable weight-tubes. These are ground to fit over conical brass pieces under the floor of the case, the upper side of which has a circular grove into which the flange f_1 f_2 round the cover is ground. All the ground joints are in general made air-tight with rubber grease, a leakage of less than 0·01 mm. of mercury a day being thus obtainable with a vacuum within the case. In certain investigations, where the use of rubber grease was excluded, air-tight joints were secured by means of seals made from an alloy of low fusibility, covered externally with low-volatile cement (as a safeguard against porosity of the alloy). Low-volatile cement is also used in case of the plate-glass window, W, through which the light from a Nernst lamp passes to the balance mirror, the image of the filament being focussed on a vertical millimetre scale at a distance of, say, 4 metres.

The shape of the beam itself, as used in my own investigations, is seen from the sketch in Fig. 1. It is made in the ordinary way on the surface of a piece of gas-coal. The fibres are drawn out from small silica-hooks fused to the beam. With a balance of this shape, 10 cm. in length, and weighing about 400 mg., a working sensibility of one millionth part of a milligram was realised with a load of 20 mg. (*i.e.*, a relative sensibility of one in twenty millions). In one series of measurements a maximum sensibility four times higher was realised, but exact weighings were then made very difficult by the tremors in the Högskola building. These tremors, due to the heavy traffic outside, are rather disconcerting, but did not sensibly affect the balance at moderate values of its sensibility.

With the newest type of fibre-suspended micro-balance of reduced size, having a beam only 5 cm. in length, maximum load between 100 and 200 mg., a maximum working sensibility of one ten-millionth part of a milligram is possible, a limit which is desirable for certain investigations.

On the other hand, fibre-suspended balances of a stronger type have also been made, which could weigh 2 grammes to less than 0·001 mg. Instruments of this type ought to prove useful for micro-chemical work.

The technique of the weighings with the fibre-suspended balances is similar to that employed with the knife-edge balances, although the operations are considerably simplified owing to improvements in the balance case.

The weighings by pressure are carried out with a bulb of known capacity, V, the change in its effective weight due to variation in the pressure and absolute temperature of the air in the case from p_1, T_1, to p_2, T_2 being found from Steele and Grant's equation,

$$dw = \delta_0 \frac{273}{760} \left(\frac{p_1}{T_1} - \frac{p_2}{T_2} \right) . V ;$$

normal density, δ_0 of the air being secured by filtering it slowly into the case through tubes containing soda-lime, barium monoxide and dry cotton wool (the last to remove dust particles). The pressure is read on an ordinary mercury manometer in communication with the case ; the temperature by a thermometer inserted into a special cylindrical tube sunk into the cover. A quarter of an hour is generally sufficient for the case to take up a uniform temperature.

When changes in weight outside the range of the air-weight are to be measured, one has to make use of silica weights standardised " against air." The lightest of those used by Ramsay & Gray weighed 0·1 mg., and were very difficult to handle. It is preferable to make instead " difference weights" for the fractions of a milligram, *i.e.*, a set of small weights of about 1 mg. each, but differing from a certain zero-piece of each set by differences of approximately 1, 2, 2 and 5-tenths, which differences are also standardised against air.

When working in a vacuum or at a constant pressure small changes in weight can be measured by deflection, whereas larger variations may be counter-balanced by a " magnetic weight " consisting of a small steel needle enclosed in silica, depending from the balance and pulled down by an adjustable magnetic field ; this field being produced by sending an electric current through a coil surrounding the weight-tube, and the strength of the current read on a milliampere-meter. The value of the "magnetic weight" for different strengths of current has, of course, also to be standardised against air.

The theory of a balance suspended by fibres affords several points of interest. It is easy to prove that when the beam swings an extremely short length of the fibres becomes curved. This vital part of the fibre needs to be very thin, generally between 1·5 μ and 2·0 μ for high-sensitivity instruments. The thickness of the rest of the fibres is unimportant. Thin as these fibres are they offer a definite, although very slight, elastic resistance to the bending forces; this increases with increasing deflection of the beam from the upright position, when the fibres are straight. This position consequently represents a *maximum* of the sensibility, which is reduced on either side of it with growing deflection. This variation, which is relatively small at moderate values of the sensibility, is of little importance as regards the technique of the weighings, since the balance is ordinarily worked at constant deflection. Moreover, as it is perfectly constant, the sensibility curve can easily be determined by means of the air-weight .

From a theoretical point of view these variations of sensibility are rather interesting, and have been made the subject of a separate study. Their amount decreases with the radius of the fibres as R^2, whereas the tensile strength of silica fibres is known to decline at a much slower rate; that is, the fibres gain much more rapidly in flexibility than they lose in carrying capacity. One may say, therefore, that the more the size of the instrument is reduced, and the finer the fibres are drawn, the *better* will this kind of suspension work, whereas the opposite is, of course, the case when knife-edges are used.

The fibre suspension presents several other practical advantages over the knife-edge. The latter is apt to get blunt with use and is easily dismounted; it often gets fouled by dust particles. The fibres retain their elastic properties indefinitely and cannot be fouled in any way.

At ordinary pressures the swing of the balance is strongly damped by air friction, but in certain experiments in a charcoal vacuum the beam was found to keep up its swinging motion without any sensible damping, the energy dissipated within the thin fibres being of course infinitesimal. Owing to the perfect properties of this suspension it should be possible to increase the accuracy of the weighings simply by reading the position of the balance very sharply, say with a microscope or by making use of increased scale distance. The sensibility curve at the end of this Paper seems to confirm this view.

Investigations with the Micro-Balance.

Of the numerous subjects of research with the micro-balance which have suggested themselves to the author, only a few have hitherto been carried out, mainly owing to the pressure of other work.

By weighing in a charcoal vacuum a small sphere of solid silica depending from the balance into a long silica tube, which could be heated up to 1,000° C. by a cylindrical electric furnace, it has been possible to repeat on a diminutive scale and at much higher temperature the experiments of Poynting & Phillips on the influence of temperature upon weight. The upper limit for any possible temperature coefficient of gravity (10^{-9} per 1°C.) found by these authors within the range $-180°$ C. to 100° C., was found to hold up to 460°C. with rather a higher degree of accuracy. When the experiments were repeated at 600°C., the silica sphere was found to undergo a gradual loss in weight at an approximately constant rate, which was much increased when the temperature was raised to nearly 800°. This rather unexpected result indicates that in a high vacuum silica is sensibly volatile even at 1,000°C. below its melting point! This substance cannot therefore be used for an extension of these experiments to still higher temperatures.

A similar or slightly modified experimental arrangement can obviously be used for investigations regarding the volatility of various substances at relatively low temperatures, and also for exact measurements of their vapour tension.

The changes in weight of a piece of gold subsequent to heating, which Ramsay and Gray had discovered and explained as due to adsorbed air, has also been studied. For this purpose a minute electric furnace (made from platinum leaf over a framework of silica) was mounted *within* the weight-tube, so that a piece of gold foil surrounded by the furnace could be heated at different pressures while suspended from the balance, and weighed immediately afterwards. The results proved that an adsorption and liberation of *air* from the gold could not be the cause of the phenomenon, but that this was due to a condensation on the metallic surface of vapours from the rubber grease used to make the case air-tight. Although the effect thus loses some of its theoretical interest, it is of a certain importance for the technique of the weighings, and will be made the subject of further research.

A subject which has hitherto been practically inaccessible to experimental work owing to the lack of sufficiently sensitive balances is the magnetic susceptibility of diamagnetic gases and vapours. A silica micro-balance affords the most favourable possibilities for such measurements, and it has also been used for determining the magnetic susceptibility of pure nitrogen and hydrogen gas. The former substance does not seem to have been at all studied by direct experiments, whereas the most recent measurement with pure hydrogen (made by Bernstein in 1909) gave a result about 100 per cent. higher than the correct value.*

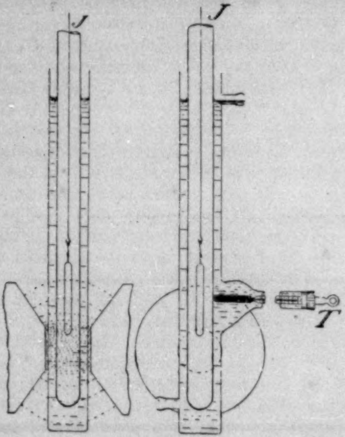

FIG. 3.

The method employed is made clear by the sketch in figure 3, which represents a very light and narrow cylindrical bulb depending from the balance, with its lower end near the centre of a strongly divergent magnetic field.

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* Bernstein: "Inaugural dissertation."

The "magnetic buoyancy" due to the gas round the bulb when the field was applied was compensated by varying the pressure. Experiments were made with air (at different pressures) and with very pure nitrogen and hydrogen gas, so that the magnetic susceptibility of the latter could be calculated from the known susceptibility of air. The results agree well with the theoretical values calculated by M. Pascal from the diamagnetism of nitrogen and hydrogen compounds, viz.—*

hydrogen $\kappa = -2{\cdot}62 \times 10^{-10}$ (theoretical value $\kappa = -2{\cdot}6 \times 10^{-10}$)
nitrogen $\kappa = -5{\cdot}57 \times 10^{-10}$ (theoretical value $\kappa = -5{\cdot}0 \times 10^{-10}$)

The susceptibility of liquid hydrogen has been measured by Kamerlingh-Onnes & Perrier, the result recalculated for hydrogen gas giving $\kappa = -2{\cdot}4 \times 10^{-10}$.†

The sensibility of the balance in itself, although moderate, would have allowed of an accuracy of about 1 per cent. That the probable error in the above results is nearly ten times larger is chiefly due to the large errors caused by temperature variations. In order to compensate for the diamagnetism of the silica in the bulb the latter had been filled with a suitable mixture of air and oxygen, so as to make it magnetically neutral at room temperature. The paramagnetism of the enclosed oxygen being a function of the temperature, this arrangement was highly susceptible to temperature changes, which could not be entirely eliminated by the water-jacket shown in the sketch. With a similar experimental arrangement modified so as to neutralise the temperature effect, the magnetic properties of other gases and vapours might be accurately measured. The author hopes especially to investigate whether when absorbing or fluorescing under the influence of light a gas or vapour has its magnetic susceptibility affected.

There are, besides the subjects mentioned, a great number of others calling for investigation with the micro-balance. Only a few of these will be here indicated.

The experiments of Nichols & Hull‡ on the pressure of light might be repeated with a much simpler arrangement, by weighing on a highly sensitive micro-balance the pressure exerted on a reflecting disc from below by an ascending beam of light. An accuracy of measurement of 1 per cent. or even less would be quite feasible, and would make possible interesting variations in the conditions of the experiment.

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* Pascal: "Ann. Chim. Phys.," VIII., 19, p. 5.

† Kammerlingh-Onnes and Perrier: "Comm. Phys. Lab. Leyden," 122A., p. 10.

‡ Nichols & Hull: "Drude's Ann.," 12, p. 225.

The ingenious method due to Prof. Knudsen* for measuring accurately very low pressures by an "absolute manometer" (by measuring the radiometric force between two discs of known unequal temperatures in a gas at low pressure) might be applied to the measurement of absolute temperatures with the aid of the micro-balance. If two horizontal discs of unequal temperatures T_1 and T_2, one of them suspended from the balance, are brought within suitable distance of each other, the radiometric force F between them could easily be determined by weighing. Supposing the pressure p of the gas in the case (say highly rarefied helium) to be accurately known, then the ratio between the absolute temperatures is given by Knudsen's equation $F=\frac{p}{2}\sqrt{\frac{T_1-T_2}{T_2}}$, and if either temperature is known the other can be calculated. By this method temperatures near absolute zero might be measured with a high degree of accuracy.

According to modern conceptions of the relationship between energy and matter the gain or loss of a certain amount of energy, dE, from a body should correspond to a change in its weight, dW, given by $dW=\frac{dE}{c^2}$, where c is the velocity of light. All ordinary changes in energy are too small to give rise to any perceptible changes in weight according to this formula, the only exception being those accompanying radioactive disintegration. If a large quantity of radium emanation, say, 2 curies, could be weighed, first fresh and then after disintegration into Radium *D*, the "weight" of the energy given off in the meanwhile should be of the order of one millionth of a milligramme, an amount which might be measured with a highly sensitive micro-balance. Owing to the intense thermal and electric phenomena accompanying the disintegration, the technical difficulties of making such weighings will no doubt be formidable, but the importance of such an "experimentum crucis" appears so considerable as to make it well worth an attempt.

The Strömberg Suspension.

The number of investigations open to micro-balances, which would add at least three or four decimals to the ordinary range of measurement, with a gain instead of a loss in relative sensibility, is obviously almost unlimited.

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* Knudsen: "Ann. der Phys.," 28, p. 75.

It is therefore rather remarkable that, except for the work of Ramsay and Gray, the quartz micro-balance remains almost unused even to-day. The reason is probably that it appears to be regarded as extremely difficult to work with, and requiring a quite unusual manipulative skill on the part of the operator. This impression had some foundation as regards the knife-edge balances and also, although to a less degree, regarding the early types of the fibre-suspended balances. Thanks to some very important improvements in construction which have lately been worked out at Stockholms Högskola, by Dr. Strömberg, these are now quite as simple to work with as any other highly sensitive physical instrument.

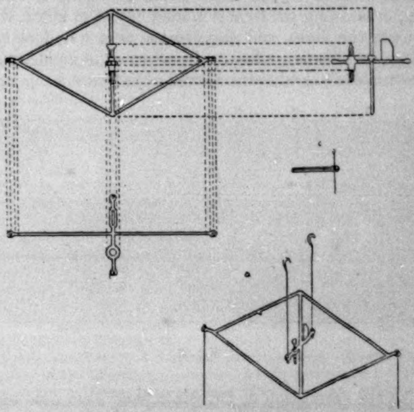

Fig. 4.—The Strömberg Suspension.

The Strömberg suspension is sketched in Fig. 4, and needs but a few words of explanation. The fibres are made detachable, with small conical lumps at one end which fit into small silica rings on the beam. The fine part of the fibre is passed through a narrow cut in the side of the ring, and the conical lump is secured in its place with a minute quantity of shellac. If a fibre should be accidentally broken a reserve fibre can be inserted in its place in a few minutes without removing the balance from its place.

In the early instruments where the fibres were drawn out from the beam, a broken central fibre could only be repaired by dismounting and inverting the beam and drawing out from it a new fibre, an operation which was rather delicate and not without a certain risk of destroying one of the other fibres. Another improvement of considerable practical value, which has also been worked out by Dr. Strömberg, is the tiny magnalium screw seen in the figure, by which the centre of gravity of the beam (and consequently also its sensibility) can be raised without difficulty. The older method of sealing on to the beam, or drawing off from it minute pieces of silica, was dangerous to the fibres, and also tedious as it was done quite at random, and therefore had to be repeated a great number of times in order to realise the desired adjustment.

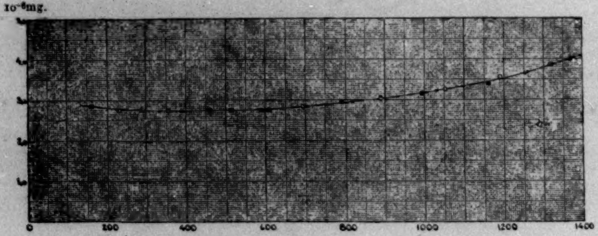

Fig. 5.

With the permission of Dr. Strömberg I am able to reproduce in Fig. 5 a sensibility curve obtained with one of his instruments (which I had the pleasure of showing to the Physical Society of London at its meeting on March 12, 1920). It is derived from three independent series of measurements, and although the scale-sensibilty is very moderate ($2{\cdot}7 \times 10^{-6}$mg. per 1 mm. scale-deflection at four metres scale-distance), corresponding to a relatively short period of swing, the different points on the curve are seen to agree to within less than one ten-millionth part of a mg. This result lends support to the view that the suspension of the instrument is in itself sufficiently delicate to allow the working sensibility to be considerably raised, merely by increasing the accuracy with which the position of the balance is observed, without any further prolongation of the period of swing.

Thanks to these improvements made by Dr. Strömberg, who will shortly publish an account of his own work, the technique of the fibre-suspended micro-balance has been much simplified, and it can now be used by anybody with scientific training. Without exaggeration one may, therefore, say that this micro-balance, which in its origin is due to the genius of British men of science, to Mr. C. V. Boys, who first invented the wonderful quartz-fibres, to Steele and Grant, to Ramsay and Gray, has by stages been developed into a working instrument which provides the chemical and physical sciences with a new and powerful weapon for future research.

ABSTRACT.

Previous Micro-balances.—Brief reference is made to the instruments of Angström, Salvioni, Nernst, and especially to the all-quartz balance of Steele and Grant.

The Author's Micro-balance.—The instrument devised by the author, and here described, is also wholly of quartz. The knife-edge support is, however, replaced by a suspension consisting of a pair of quartz fibres. A magnetic arrestment is employed. Balance is obtained either (*a*) by an "air-weight," the upward displacement on a bulb being varied by varying the gas pressure within the enclosing case; or (*b*) by a "magnetic weight," involving the use of a measured electric current. The range of weighings depends on the size of beam: with one 5 cm. in length the maximum load is between 100 and 200 mg., and the setting can be made to within 10^{-7} mg.

The Strömberg Suspension.—Improvements in the construction devised by Dr. Strömberg are described, which obviate the formidable difficulties of manipulation that have hitherto discouraged workers from employing the instrument.

Investigations with the fibre-suspended Micro-balance.—A brief description is given of the following researches already carried out:—(1) The influence of temperature upon weight. This is an an extension of the work of Poynting and Phillips on this subject; (2) the volatility of silica; and (3) the susceptibility of diamagnetic bodies.

It is proposed to conduct investigations on the pressure of light, on the measurement of absolute temperature (based on an equation of Knudsen's), and on the loss of mass accompanying loss of energy.

XXII. *A Modified Theory of the Crookes Radiometer. By* GILBERT D. WEST, M.SC. (LOND.).

A striking feature of the physical literature of the period 1874-1881 is the extraordinary amount of interest that was raised by Crookes' researches "On the Repulsion Resulting from Radiation." In addition to Crookes himself, Fitzgerald, Johnstone-Stoney, Stokes, Schuster, Pringsheim, Osborne-Reynolds, and Maxwell, all gave the phenomena their attention. It was the work of Osborne-Reynolds and Maxwell, however, that was held to settle the numerous discussions that had arisen. These physicists showed, in two long and highly mathematical Papers, published in the Phil. Trans. of 1879, that the phenomena under notice were explicable in terms of the then somewhat new kinetic theory of gases. As a result of their Papers, the interest in the theory of the radiometer—great though it had previously been—subsided, and but for a criticism of Reynolds' work from Fitzgerald in 1881,* there is little worth recording for the next sixteen years. It would seem that this change occurred, largely because the majority of physicists felt that the matter had been placed beyond their grasp, rather than because some well-understood explanation had been given which rendered further research superfluous. According to Fitzgerald, Reynolds had rendered "a difficult subject tenfold as elaborate as was necessary," but however this may be, it is certain that an easily intelligible explanation of radiometric phenomena was wanted. Attempts were made, it is true, to give the substance of the accepted theories in general language, but, although many of the text book "explanations" are still based on these, it cannot be said that they met with much success.

In 1896, however, Sutherland turned his attention to the subject, and his Paper† had the advantage that, unlike those of Reynolds and Maxwell, it could be read with profit by the physicist of ordinary mathematical attainments. Yet appearing as it did at a time when the interest in the theory of the radiometer had practically ceased, it was neglected, and has since become almost entirely forgotten.

It was shown by the author,‡ however, that Sutherland's theory of the closely related phenomena of thermal transpiration was so far correct, that it could be made to form an adequate basis for the careful experimental work of the Danish physicist, Knudsen, performed thirteen years later. The author was thus tempted to place considerable confidence in Sutherland's theory of the radiometer. It must be observed, however, that, at the time when Sutherland wrote his paper, little was known of the thermal surface conditions in a rarefied gas, and thus it is not surprising to find that Sutherland's theory —like many another imperfect theory in other branches of physics—will explain some of, but not all the experimental facts. It is the object of the present Paper to outline a theory of a general descriptive character which, while being to a large extent based on Sutherland's work, makes use of knowledge gained since that time.

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* Phil. Mag., 11, p. 103, 1881.

† Phil. Mag., 42, p. 373 and p. 476, 1896.

‡ Proc. Phys. Soc., Vol. XXXI., p. 278, 1919.

Sutherland, like Reynolds, bases his theory of the radiometer on the closely related phenomena of thermal transpiration. He first considers the case of a tube along which a temperature gradient is maintained. If the tube connects two infinite spaces, he shows that an even flow of gas takes place over the whole cross section of the tube from the hot to the cold side. The walls of the tube in thus constraining the gas to take their temperature, exert a tangential traction upon it, whilst they themselves experience a reaction in the opposite direction. A nett reaction is only experienced, however, near the entrance of the tube, where the velocity of the gas rises from zero to the uniform value eventually attained. At all other parts of the tube the friction of the moving gas against the walls of the tube exactly balances the traction the walls exert on the gas; it is to the "unequilibrated traction," however, that Sutherland attaches most importance.

If the spaces the tube connects are finite, the uniform flow of gas produces a pressure in the hot vessel, which in turn produces a flow of the Poiseuille type in the reverse direction to the original flow. The result of the superposition of these two flows is to give a gas current from the cold to the hot side along the walls of the tube, together with a current from the hot to the cold side along the axis, whilst between the two currents there is a surface of zero velocity.

With reduction of gas pressure, the Poiseuille counter flow becomes less and less important, until at the highest rarefactions it is negligible, and the hot regions are then enabled to maintain undiminished their higher pressures.

Sutherland points out that both the "unequilibrated traction" and the thermal transpiration pressure operate in causing radiometer motion, and to make his meaning clear he takes the case of a piston that fits loosely into a cylinder closed at both ends. He imagines the temperature of one compartment to be higher than that of the other, and that a temperature gradient exists in the material of the piston. As a consequence of the operation of thermal transpiration, the pressure on the hot side will become greater than that on the cold side, and a force will be exerted on the piston, both as a result of this excess pressure in the hot compartment, and as a result of the unequilibrated traction on the sides of the piston. By increasing the clearance between the piston and the cylinder, the thermal transpiration pressure can be made to bear an ever-decreasing ratio to the total thrust on the piston. If the piston be free to move, it constitutes, according to Sutherland, an exaggerated example of radiometer motion, and he formulates his theory on these lines.

It must be admitted, however, that to pass from the case of the piston to that of the vane of a Crookes radiometer, is a big step. Nevertheless, Sutherland maintains that such a step is legitimate and proceeds, with considerable success, to analyse the experimental data obtained with a type of torsion radiometer. Except in so far as it was better suited to give quantitative measurements, this type did not differ materially from the familiar type in which flat circular vanes, blackened on one side and silvered on the other, are mounted at the ends of a pivoted arm. "When the black face is irradiated," he says, "there is a fall of temperature . . . through the thickness of the vane, and thus the thickness of the vane becomes a surface capable, along with the surface of the bulb opposite it, of starting thermal transpiration from the cold edge to the hot, with elevation of pressure in front of the hot face . . . and depression of that behind the cold face."

Such, briefly, is an outline of Sutherland's theory, but it is clear, as indeed he himself states, that it is based on the idea that the effect of the introduction of a heated body into a gas is "to make the layer of gas in contact with the solid take the temperature of the solid at every point of the surface." It has since been realised, of course that this is not so, and that in any calculation made in regard to rarefied gases, due allowance must be made for the temperature discontinuity at the surface of the solid.

To make what is meant more clear, consider the case of two large parallel plates—one hot and the other cold. We may regard the molecules that strike the hot surface, both as coming from a distance of the order of the mean free path, and as possessing the temperature of this region. If, after reflection at the surface, these molecules merely acquire its temperature, it is clear that the mean temperature of the surface layer of gas is necessarily lower than that of the surface itself. As a matter of fact, however, it has been shown* that, on the average, impinging molecules do not even acquire the temperature of the surface they strike, and hence the discontinuity already noted becomes enhanced.

Thus in the light of modern research, one of Sutherland's assumptions is seen to be unjustifiable, and the theory built upon it must therefore be imperfect. It is possible, however, to modify the theory.

Consider once again the case of the hot and cold plates, but now with reference to the change that takes place in the isothermals, as the pressure is lowered. It is clear that, with increasing mean free path, the extreme isothermals will disappear, and that eventually, when the mean free path becomes large compared to the distance apart of the plates, the gas between will reach a uniform mean temperature. Reduction of gas pressure will thus be accompanied by a gradual diminution, and eventual elimination, of the temperature gradient between the plates.

In the particular case considered, it is easy to calculate the positions of the isotherms with changing gas pressure, but in the case of a radiometer vane this is much more difficult. However the present purpose is served quite well if a general indication is given of the changes that take place.

To make matters simple, consider first of all the case of a disc with the hot surface A 10° C. above the cold surface B—the walls of the containing vessel being supposed distant, and at the mean of the temperatures of the hot and cold surfaces. In Fig. 1 the isotherms in the gas are drawn for the case where the gas pressure is very high, whilst in Fig. 2 a rough idea is given of their appearance at a lower pressure. It will be seen that the extreme isothermals have disappeared, and that, so far as the gas is concerned, neither surface of the disc coincides with an isothermal surface. On the contrary, a temperature gradient extends from the circumference towards the centre.

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* Soddy & Berry, Proc. Royal Soc., A, 84, p. 576, 1911; Knudsen, "Ann. d. Phys.," 34, 4, p. 593, 1911; Smoluchowski, "Phil. Mag.," XXI., p. 11, 1911; Langmuir, "Phys. Rev.," Vol. II., No. 5, p. 329, 1913.

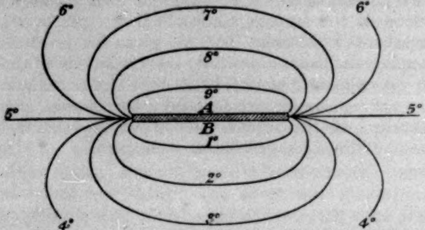

Fig. 1.—Isotherms Around Radiometer Vane—Pressure High.

Such a temperature gradient in the surface layer of gas was encountered in the experiments described in a previous paper by the author on the forces acting on heated metal foil surfaces in rarefied gases,* and it was there shown that satisfactory explanations could be based on the assumption that this temperature gradient gave rise to gas currents somewhat similar to those arising in a tube in the material of which a temperature gradient was maintained in the direction of the length.

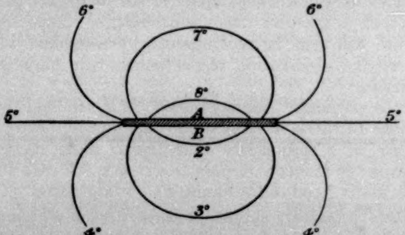

Fig. 2.—Isotherms Around Radiometer Vane—Pressure Low.

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* "Proc." Phys. Soc., Vol. XXXII., 1920.

If, as would appear legitimate, we can apply similar reasoning to a radiometer vane, we shall have to imagine a flow of gas to take place similar to that indicated in Fig. 3. On the cold side B, the gas will be flowing outwards from the centre, with consequent reduction of pressure on the vane, whilst on the hot side A it will be flowing inwards towards the centre, with consequent increase of pressure on the vane. When the gas pressure is fairly high, and when the isotherms have consequently not departed much from the shapes indicated in Fig. 1, the flow of gas and the regions of excess and defect of pressure will be restricted to the peripheral portions of the disc. When, however, the gas pressure is lowered, the area over which the excess or defect of pressure is felt, will extend towards the centre, until at last the whole surface of the disc will be covered.

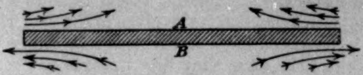

FIG. 3.—FLOW AROUND RADIOMETER VANE.
A is Hot Surface, *B* is Cold Surface.

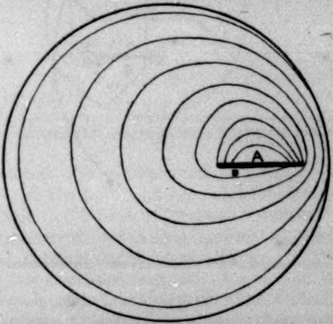

FIG. 4.—ISOTHERMS AROUND RADIOMETER VANE IN SPHERICAL GLASS BULB—MEDIUM GAS PRESSURE.*
A is Hot Surface, *B* is Cold Surface.

The shapes of the isotherms surrounding an actual radiometer vane enclosed in a spherical bulb are, of course, again far too complicated to calculate. From general considerations, however, it is possible to give a rough idea of their shapes. In practice, one side of the vane, although much cooler than the other, is still slightly above the temperature of the glass walls, and under such conditions Fig. 4 might represent sufficiently well the forms the isothermals would take at a medium gas pressure. An indication is also given in Fig. 5 of the directions in which the gas currents would flow. It will be seen that on both sides of the vane they flow towards the central region, but that on the hot side the flow is much more vigorous owing to the greater temperature gradient. It will be noticed further that the flow is greatest on the edge of the disc nearest the glass, and also that in the gas layer on the glass itself there are two regions each capable of producing excess pressures—the most effective region being near the hot side of the disc.

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* The suggestions made by Mr. F. J. W. Whipple in the discussion are embodied in this figure.

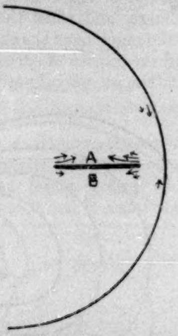

Fig. 5.—Flow Around Radiometer Vane in Spherical Glass Bulb. *A* is Hot Surface, *B* is Cold Surface.

Briefly, therefore, the modification introduced by the present explanation consists chiefly in a difference in the temperature ascribed to the surface layer of gas covering the vane, and in a consequent difference in the regions over which the flow of gas is supposed to take place. The question naturally arises, therefore, as to whether there are any radiometric phenomena that support the present view. For a reply we must turn our attention to the Crookes radiometer with slanting vanes.

In this instrument, a plan of which is shown in Fig. 6, a number of vanes are placed at an angle to the supporting radial arms. In one instrument the vanes were made of aluminium foil bright on both sides, and when radiation was allowed to fall upon them, vigorous rotation ensued (indicated in Fig. 6 by the arrow), and that in spite of an almost negligible difference in temperature of the two sides of the vanes. Crookes points out * that had the vanes pointed radially there would have been practically no tendency to rotation.

On Sutherland's theory, no very obvious explanation of this phenomenon can be given, but on the present theory this is at once possible.

The surfaces of the vanes will be covered on both sides by a gas layer whose temperature will rise from the edge towards the central regions. The gradient will be steepest on the part of the disc nearest the bulb, and here also the flow of gas will be most vigorous. We have to remember however that the gas currents can only flow as a result of a tangential reaction exerted on the material of the vane. Hence on both its surfaces we shall have outward tangential forces, and such forces will be capable of producing rotation. The side of the vane nearest the glass, moreover, will experience a pressure resulting from the difficulty the surface currents find in escaping, but this pressure will not in general be as important as the tangential forces.

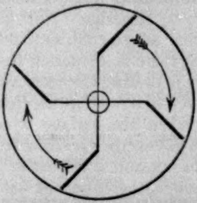

FIG. 6.—CROOKES RADIOMETER WITH SLANTING VANES.

The radiometer with slanting vanes connects itself naturally with the recent experiments of the author† on the edgewise movements of strips of foil placed at an angle to a glass

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* Phil. Trans., 169, p. 282, 1878.

† Proc. Phys. Soc., Vol. XXXII., 1920.

wall, and exposed to the radiation from a lamp. It will be remembered that such strips showed edgewise movements towards the glass which completely masked the normal repulsion effect.

Let us now pass to the consideration of the radiometer constructed of hemispherical cups, mounted anemometer fashion. Sutherland says: "Consider a convex vane irradiated by a source on the normal through its middle point; then as the amount of heat that a surface absorbs depends on its obliquity to the incident radiation, the farther a part of the convex surface is from the middle, the less is it directly heated, and thus there is a continuous fall of temperature from the centre of the surface to the edge; conduction, if allowed time, tends to reduce the amount of fall but does not obliterate it, and conduction also establishes a fall of temperature along the back from the centre to the edge. Now the traction of the gas on the solid, is from hot to cold, so that both on the front and back of the vane there is a traction from centre to edge, whose resultant effect is to drag the vane away from the light when the vane is convex to it, so that the light appears to repel a convex surface. When the surface is concave the same reasoning applies, the gas exerts a traction from centre to edge, and therefore the light appears to attract it."

According to the above, therefore, the temperature gradient is made to depend on the changing obliquity of the surface of the cup. Rotation still takes place, however, when the obliquity is kept constant by replacing the hemispheres by hollow cones. It is true that, owing to a spreading of the stream lines more heat will here be conducted away from the peripheral regions of the cone than from other regions, but with cups of good conducting material, such as aluminium, the temperature gradient so produced would be very small. Previous experiments by the author have demonstrated, moreover, that such temperature gradients are not of much importance. For the rotation of the conical cups, therefore, Sutherland's theory gives no easy explanation.

If we realise, however, that a rarefied gas does not necessarily take the temperature of the surface with which it is in contact, and if we imagine, as is legitimate, that a temperature gradient, in the gas layer, extends from the periphery of the cone towards the apex, there is no difficulty in modifying Sutherland's original explanation to suit this new case. Such a modification will resemble that given for the radiometer with slanting vanes.

In his Paper Sutherland deals with other experiments performed by Crookes, but the differences that have to be introduced into his explanations on the present theory are so small that further discussion is not necessary.

It might be said, therefore, when all the facts that have been discussed are taken into consideration, that Sutherland's theory, as it stands, is apparently not capable of explaining certain radiometric phenomena, but that when modifications, supported by modern experimental knowledge, are made in the thermal surface conditions, such explanations become at once possible.

ABSTRACT.

The Paper gives a short account of a theory of the Crookes Radiometer worked out by Sutherland in 1896, but unfortunately since much neglected. The theory as it stands will not explain many radiometric phenomena, but it is shown that when modifications, depending on the modern knowledge of thermal surface conditions, are made, such explanations become possible.

Radiometer action, especially at the higher gas pressures, would appear to depend essentially on the formation of gas currents near the radiometer vane. These currents are distinct from convection currents, but are closely connected with the phenomena of thermal transpiration.

The present work is the natural outcome of previous work by the author (Proc. Phys. Soc., Vol. XXXII.).

DISCUSSION.

Mr. F. J. W. **Whipple** pointed out that as the isothermals were drawn in Fig. 4, they indicated that the colder side of the vane was actually colder than the walls of the bulb, whereas actually it was hotter. He indicated alterations in the shape of the isothermals which would correct this. In connection with the general presentation of the problem he would have liked it better if it had been based more on kinetic theory and less on hydrodynamical flow. It was necessary to introduce kinetic theory to explain the difference of pressure between the hot and cold ends of a tube. When the mean free path was large, a particle entering one end of the tube would strike the sides tangentially and pass through the tube. When equilibrium was reached the number of particles entering at each end must be equal, so that $\rho v=\rho' v'$. But the condition for equality of pressure was $\rho v^2=\rho' v'^2$.

Mr. **West**, in reply, said he was not quite clear at the moment on the point raised about Fig. 4. He would look into it and communicate a reply.* In an earlier Paper he had pointed out what Mr. Whipple had just mentioned, that at low pressures it was the conservation of matter that determined equilibrium, while at high pressures it was the equality of pressures.

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* See footnote to page 227.

XXIII. *On the Magnetic Properties of Silicon Iron (Stalloy) in Alternating Magnetic Fields of Low Value.* By ALBERT CAMPBELL, *B.A.* (*From the National Physical Laboratory.*)

RECEIVED FEBRUARY 28, 1920.

1. *Introduction.*

THE behaviour of magnetic materials, and in particular of silicon iron, at very low magnetisations, is of importance in telephony and radio telegraphy, as telephones and transformers in general have iron cores. The following investigation was carried out in 1916 and 1917 for the Board of Invention and Research, and is now published by the kind permission of the Admiralty. The particular silicon iron tested was Stalloy made by Messrs. J. Sankey & Son.

As the hysteresis loss, with which we are here chiefly concerned, falls off more rapidly than the square of the flux-density **B**, it becomes so small at low magnetizations that it is difficult to measure it with accuracy. In Prof. E. Wilson's experiments,* to which we shall allude further below, the lowest value of **H** used was 0·0036. We succeeded in making measurements down to about $\mathbf{H}_{\text{max.}}=0{\cdot}0002$. For this low value the total amount of power measured was of the order of one millionth of a microwatt. In addition to determinations of power loss both at low and telephonic (audio) frequencies, the difference in properties of wire and sheet samples was examined, and investigation was made of the effect of superimposing on the small alternating field a much larger direct-current steady field.

2. *Hysteresis Loss at Low Frequencies.*

The silicon iron in the principal tests was in the form of ring stampings of Messrs. Sankey's stalloy sheet about 0·4 mm. thick, insulated on one side. A number of these had been made up into a ring of mean diameter 7 cm. and section 3 sq. cm., with 10 windings of 10 turns each and 4 of 200 turns each, being, in fact, one of our small laboratory transformers.

With 10 primary turns and 800 secondary, various ballistic tests were made, but the sensitivity was not sufficient below values of **H** of about 0·02. Accordingly I had recourse to an alternating-current method which is very much more sensitive, and had been found very convenient for similar measurements of the power lost in the iron wrappings of loaded telephone cables. I have already described the method (Proc. Phys. Soc., p. 24, Vol. 22, 1910), but a brief description here will make matters clearer.

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* Proc. Roy. Soc., p. 548, Vol. 80, June, 1908.

The iron ring is wound with superimposed primary and secondary turns N_1 and N_2 respectively, m being the mutual inductance between them. They are connected, as shown in Fig. 1, through the coils of a mutual inductometer D to a source of alternating current and a vibration galvanometer (G), a sliding contact E allowing a portion Q to be selected from a slide-wire or other suitable resistance connected to the junction of the coils of D. Let the primary current I_1 (by tuning or otherwise) be of sine wave form, and let the galvanometer be tuned to its frequency. The galvanometer deflection can now be reduced to zero by adjusting D and E to values M and Q respectively. When the eddy currents in the ring are small we have

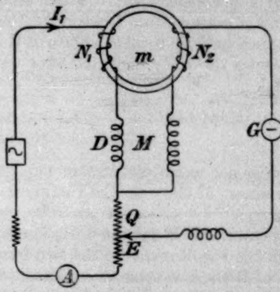

FIG. 1.

$$m = M,$$

and total iron loss

$$= \frac{N_1}{N_2}, QI_1^2.$$

Since $m = 4\pi \times 10^{-9} N_1 N_2 \mu s / l$, where μ is the permeability, s the section and l the mean circumference of the ring, we can at once find the permeability μ for various values of I_1 (and hence of **H**). The value of **H** is deduced from $I_1\sqrt{2}$, where I_1 is the effective value of the sine wave current (measured by the ammeter A).

The low-frequency tests were made chiefly at 10 and 75 ∿ per second. At 10 ∿ per second the oscillations of the light spot from the vibration galvanometer are easily visible, and extreme accuracy of reading can be obtained by adjusting until no tremor is perceptible.

With $\mathbf{H}=0{\cdot}004$ at 10 ∿ per second, M (and hence μ) could be read with a sensitivity of 1 in 10,000, while the loss could be observed to 1 or 2 per cent. At 75 ∿ per second the eddy current loss forms a considerable fraction of the total power loss, being more than half the total at values of **H** below 0·0004. For both frequencies the eddy current losses were calculated from the formula

$$h' \doteqdot 1{\cdot}64 \, \frac{(2a)^2 n^2 \mathbf{B}^2_{\text{max.}}}{\sigma},$$

where h' is in ergs per cubic centimetre per cycle, σ is the resistivity (here about 57,000), $2a$ the thickness of the sheet, and n the frequency.

For the upper values of **H** (about 0·01) the hysteresis loss thus derived is in good agreement for the two frequencies. At the lower range of **H** there is some divergence, the higher frequency giving the smaller values. In any case the eddy current loss is of small importance at 10 ∿ per second, and the result which we proceed to give has therefore been deduced only from the observations at this frequency.

Bearing in mind that μ remains nearly constant over the whole range of **H** dealt with, we deduce $\mathbf{B}_{\text{max.}}$ from $\mathbf{B}=\mu\mathbf{H}$. If h be the observed hysteresis loss (in ergs per cubic centimetre per cycle), when log h was plotted against log **B**, the resulting curve was found to be nearly a straight line. The results are quite well represented by the two following equations :—

H range.		**B** range.		Equation.
0·0002 to 0·002	……	0·051 to 0·51	……	$h \doteqdot 6{\cdot}0 \times 10^{-6}\, \mathbf{B}^{2\cdot 3}$
0·002 to 0·02	……	0·51 to 5·1	……	$h \doteqdot 6{\cdot}6 \times 10^{-5}\, \mathbf{B}^{2\cdot 40}$

These results refer only to the particular sample tested, for which the permeability μ over the range of **H** employed is about 254.

Prof. E. Wilson's ballistic tests on stalloy (of initial permeability about 260) for a range of **H** from 0·004 to 0·04 give

$$h \doteqdot 6{\cdot}9 \times 10^{-6}\mathbf{B}^{2{\cdot}30}.$$

Our own ballistic tests from $\mathbf{H}=0{\cdot}015$ to 0·1 gave

$$h \doteqdot 14 \times 10^{-6}\mathbf{B}^{2{\cdot}4};$$

but magnetic viscosity effects somewhat complicate tests with slow cycles.

It appears that as **H** is lowered the index of **B** decreases slightly. I have little doubt that the constants will vary very considerably from sample to sample, but I think that the equations given represent the properties of good material ($\mu_0=254$). It may be mentioned that the values of the permeability obtained at 10 and 75 ∿ per second were almost identical (within 4 parts in 1,000), rising from 254 at $\mathbf{H}=0{\cdot}00008$ to 264·5 at $\mathbf{H}=0{\cdot}01$.

3. *Power Loss at Telephonic Frequencies.*

By the same method, but using a telephone in place of the vibration galvanometer, similar tests were made both at 500 and 800 ∿ per second, the values of the applied $\mathbf{H}_{\text{max.}}$ running from 0·0002 up to 0·1. When the logarithm of the total power loss was plotted against log $\mathbf{B}_{\text{max.}}$ the resulting curve was in each case an approximate straight line, the line for 500 ∿ per second being parallel to but somewhat below that for 800 ∿ per second. The hysteresis loss was calculated by the same formula as before, assuming uniform flux density in the material. The (log **B**, log h) curves at the two frequencies come to nearly the same straight line. This line is parallel to, but considerably above, the line given at 10 ∿ per second, showing the effective hysteresis loss at 500 ∿ per second to be about three times that at quite low frequencies. Whether this difference is due to magnetic viscosity, or to want of uniformity of **B** throughout the section of the material (skin effect), or to some omitted correction, is not clear; the point requires further elucidation.

In some of the experiments the material appeared to show a slightly lower initial permeability (*i.e.*, for $\mathbf{H}=0$) at the higher frequencies, but this was probably due to effects of previous history which are discussed in §5 below.

4. *Differences between Wire and Sheet Samples.*

With stalloy as supplied commercially we have noticed that various samples in the form of wire do not show nearly such good magnetic properties as the sheet material. This is unfortunate, as wires have advantages for certain types of core. The cause of the difference was investigated, and it appears to be due to insufficient annealing of the wire. Various samples of wire were annealed in hydrogen (or nitrogen) at a temperature of about 800 °C. Some were tested in ring form, others in long thin bundles placed (with a central search coil) in the middle of a long magnetizing solenoid. In all cases for the lower values of **H** the mutual inductance method was used. The results for the long bundles were corrected for the demagnetizing field due to the ends. The main effect of the annealing was to increase the permeability; the power losses for the same value of **B** did not show any marked improvement. Some of the results for permeability are shown in Fig. 2. The wires marked *P* and *Q* had diameters of 0·15 mm. and 0·40 mm. respectively.

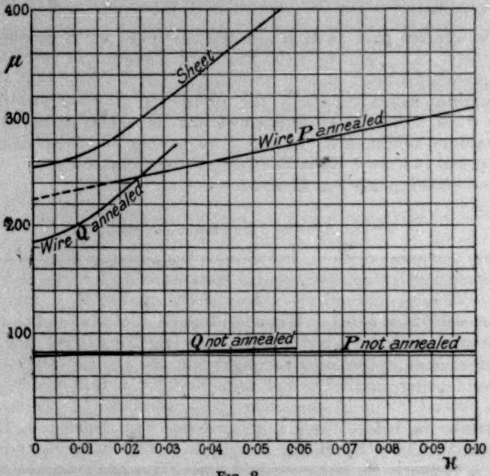

FIG. 2.

It will be noticed that the unannealed wires show very constant permeability for small values of **H**. The thicker wire (*Q*) behaves similarly to unannealed Swedish iron as given by Lord Rayleigh, for which $\mu=81+64\mathbf{H}$. The thinner wire (*P*) shows much greater constancy in permeability. The annealing makes a very great improvement in the wires, (*P*) annealed being almost as good as the sheet material. It is interesting to notice that it still retains a slower rate of increase of permeability for change of **H** than is shown by (*Q*) annealed. The wire (*Q*) after annealing is very like the very soft Swedish iron for which Ewing gives $\mu=183+1382\mathbf{H}$.

It is clear that for practical purposes, except where constancy of μ is important, silicon iron wires should be carefully annealed.

5. *Effect of Superimposed Direct-Current Field.*

In many cases in practice a core of magnetic material used with alternating magnetization may at the same time be subjected to a direct-current field; for example, in a telephone transformer used with a microphone receiver.

It is of interest therefore to investigate the effect of such a superimposed field upon the behaviour of the magnetic material as regards the alternating field. For this purpose tests were made on the Stalloy sheet and on the unannealed wire (*P*), both with ballistic galvanometer to represent quite slow frequencies (say, 2 ~ per second), and also at a frequency of 800 ~ per second. As a preliminary to these tests the ordinary permeability curve was determined for each of the materials. As will be seen from the μ curves in Fig. 3 for sheet and Fig. 4 for unannealed wire, the sheet material shows much higher permeability throughout than the unannealed wire.

6. *Ballistic Tests with Small Superimposed Cycles.*

In order to examine the effects at quite low frequencies, ballistic tests were made as follows. The material was first demagnetized by repeated cycles of gradually diminishing **H**, and then brought into a steady cyclic state by many reversals of a given $\mathbf{H}_{\max}$. The corresponding $\mathbf{B}_{\max}$ is known from the permeability curve already determined. When in the condition ($\mathbf{H}_{\max}$, $\mathbf{B}_{\max}$), the **H** is suddenly reduced by a small amount $\Delta\mathbf{H}$ by the introduction of a suitable resistance into the magnetising circuit. By cutting this resistance out and in, a small cycle of amplitude $\Delta\mathbf{H}$ is superimposed on the main field $\mathbf{H}_{\max}$.

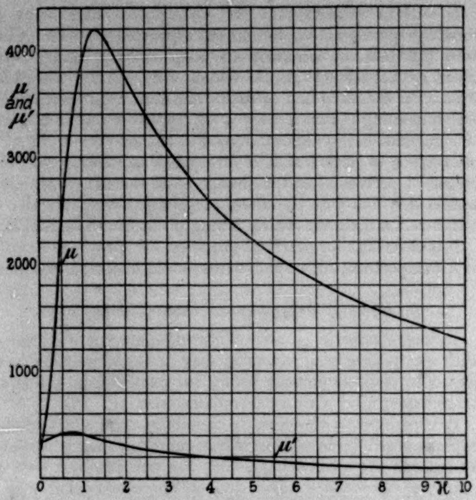

FIG. 3.—SHEET MATERIAL.

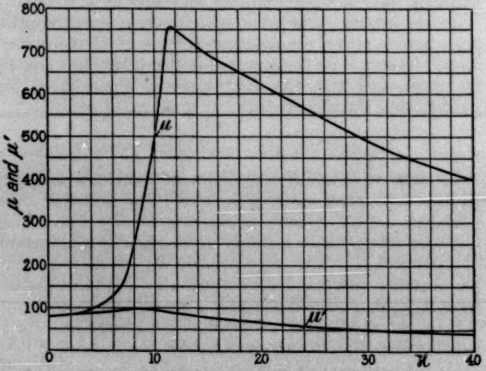

FIG. 4.—UNANNEALED WIRE.

This cycle was repeated 20 or 30 times to approximate to a steady state, and then the $\Delta\mathbf{B}$ corresponding to $\Delta\mathbf{H}$ was determined by the throw on a sensitive ballistic galvanometer.

In general $\Delta\mathbf{H}$ was not more than from 0·1 to 0·02 of $\mathbf{H}_{\max}$. The ratio $\Delta\mathbf{B}/\Delta\mathbf{H}$ was taken as the effective permeability μ' for the small cycle. The values obtained are plotted against **H** (the main field) in Figs. 3 and 4. At the lower values of **H** (and where the **H**, μ curve is rising steeply) the previous history of the material has large effect on μ', and the experimental results are a little uncertain. For each material, however, the μ' at the lower values of **H** is distinctly approaching the initial permeability μ_0 got by the more direct methods.

An examination of the curves shows that, for small slow cycles, the effective permeability μ' is considerably increased as the main field is raised. But before **H** has reached the region of maximum μ, μ' has begun to fall off again, and, as **H** goes on increasing, it soon falls below its initial value. There is every reason to believe that when **H** has risen to saturation value μ' will have fallen to the value unity. It is evident that the presence of direct-current magnetization (above the maximum permeability point) may produce very serious reduction in the effective permeability for small cycles. The effect for silicon iron turns out to be very similar to that shown for ordinary hard and soft iron in the experiments of the late Lord Rayleigh.*

7. *Tests at Telephonic Frequency.*

The mutual inductance method is easily adapted to measurements with superimposed direct-current magnetization, which we may designate the "main field." The main field and the small alternating field can be measured quite independently, as shown in Fig. 5.

The part comprising the iron ring G, the mutual inductometer M, the resistance Q (with potential slider) and the telephone T is the same as before, with the addition of the condenser C in the telephone circuit to prevent direct current from getting through the secondary winding of G round the telephone circuit. The small alternating current I_1 passing through the primary winding of G is measured by means of the thermoammeter A connected through the air core transformer *a*, the combination being calibrated for the frequency used, which was 800 ∿ per second. The source of the alternating current is an electric trumpet buzzer *Z* linked to the circuit by the variable mutual inductance *X*. This transformer does not contain iron, in order to ensure that direct current through its secondary coil shall not affect the alternating component. *K* is an adjustable condenser to give approximate tuning, so that the magnetizing current I_1 shall be nearly of sine wave form.

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* Phil. Mag., 1887. Reference ought also to be made to Prof. E. Wilson's experiments on the magnetization of stalloy when screened from the earth's field. (Proc. Roy. Soc., Vol. 90, 1914, and Vol. 93, 1917.)

The direct-current component I_0 is introduced by switching from *D* to *E*, thereby bringing the battery *b* into the circuit. I_0 is measured by the Weston voltmeter *Y* with low resistance shunt, giving readings that are entirely unaffected by the presence of the alternating component. In general the alternating component was kept of the order of 3 to 7 milliamperes. For the sheet material this gave cyclic values of $\mathbf{H}$ from ±0·006 up to ±0·011. The applied direct-current field $\mathbf{H}_0$ could be varied from 0·06 up to 2·0. Usually $\mathbf{H}_0$ was reversed eight to ten times after any change, in order to attain a fairly cyclic condition.

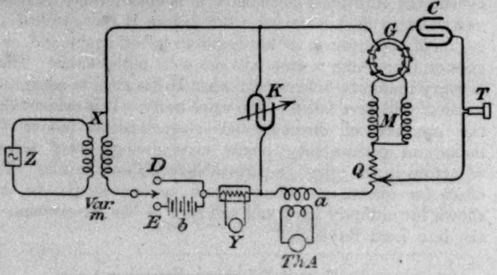

FIG. 5.

The effect of the superimposed field on the permeability μ' was quite marked. As will be seen from the following table (for the sheet material), the changes in μ' at 800 ∿ per second are very similar to those shown in Fig. 3 for very slow small cycles.

Very distinct time effects were observed. With the direct-current field, the value of the effective permeability gradually altered with time, sometimes rising slightly and sometimes

TABLE.

Time conditions of observation.	$\mathbf{H}_0$, direct-current field.	μ' (for small cycles at 800 ∾ per sec.).
	0·0	260
Immediate	0·18	278
"	0·50	313
"	1·00	275
"	2·09	196
One minute after direct current had been switched off	0·0	247
After standing for 20 hours	0·0	225

falling, according to previous history and the position on the ($\mathbf{H}_0\mathbf{B}$) curve. When the direct-current field was removed, there was also a similar creeping effect, fairly quick at first (*e.g.*, 4 per cent. change in a few minutes), and observable for several hours afterwards, even when the small alternating field was also removed, except for each moment of test. This latter creeping effect appeared always to bring the condition of the material backwards along the ($\mathbf{H}_0\mu'$) curve in Fig. 3. It did not appear to be due to tremors, for the application of fairly vigorous knocking did not seem to have much effect. The slow changes in μ' were easily observed, as the method is sensitive to 2 or 3 parts in 10,000. The effective permeability of the unannealed stalloy wire also showed similar alterations due to direct-current field, and these were in general agreement with the μ' curve of Fig. 4.

In all the experiments the total power loss was measured simultaneously with the permeability μ'. It was found that the power loss was in general altered by the imposition of the direct-current field, but the whole of this change was accounted for by the observed change in permeability.

As already mentioned the hysteresis loss for small values of $\mathbf{H}$ is practically the same in the unannealed as in the annealed material. In this connection it is interesting to mention that Mr. N. H. Williams ("Physical Review," New Series, Vol. 1, May, 1913) has observed that for small cycles of $\mathbf{H}$ (such as are here considered) there is little difference in the hysteresis losses for soft iron and glass hard steel.

In conclusion I would express my thanks to the Admiralty for kind permission to publish the above research, and to Mr. D. W. Dye for kind help in some of the experiments.

ABSTRACT.

Measurements are described of the hysteresis-losses in silicon iron sheet and wires in very low alternating magnetic fields at low and telephonic frequencies using an alternating current method described in a former Paper. The equations giving the hysteresis losses as a function of $B_{\max}$ are deduced in the case of the sheet material at low frequencies for ranges of $H_{\max.}$ from 0·0002 to 0·02.

Comparisons are made between sheet material and wires of different diameters, and curves are given showing the great improvement in the permeability of wires when they are annealed.

The behaviour of the material is studied, both by ballistic tests and at telephonic frequencies, as regards the alternating field when direct current fields of various values are applied at the same time.

DISCUSSION.

Mr. POLLOCK said he was much interested in the method of measurement and would try it at the first opportunity. He had usually employed modified Bridge methods and had got down to magnetising fields of 0·02, but not so far down as the author. He was interested in the formula for eddy current losses. He had frequently found in measurements with fine wire cores that the eddy current loss takes jumps due to the E.M.F.s set up being sufficient to make the currents pass from one strand to another, and he thought it was impossible to calculate the eddy current losses by any formula. It was important in practice when measuring the losses in stranded cores to take account of the value of the magnetic field to which the core would be subject in use, for with some fields the eddy currents would pass from strand to strand, while with others they would not. He was interested in the conclusion that hysteresis loss was proportional to the square of the flux density. He had had the impression that the loss became very small at very low fields. With regard to Fig. 2, showing the permeabilities for unannealed and annealed stalloy, it was most important in telephone work that the cores of the loading coils should be of constant permeability over the range of field to which they were subject in use. Unfortunately such cores were liable to damage by excessive magnetisation due to accidental currents flowing through the coils, as their permeability did not return to its initial value. Many of the old coils had lost about 15 per cent. of their inductance from such causes. This could only be recovered by re-annealing and unfortunately annealed stalloy had not constant permeability. On this account the Americans were beginning to use granular cores.

Mr. RITTER asked if the windings were absolutely dry, as he had found that a trace of moisture in the dielectric caused quite appreciable losses. He emphasised the importance of constant permeability in the iron used for loading coils. When a cable was loaded the impedance depended on the square root of the ratio of the inductance to the capacity per unit length. If the impedance altered with the current, difficulties were introduced, especially since the use of amplifiers became general.

Mr. DYE said the greatest care was always taken in winding the coils to prevent the possibility of moisture being present. As regards the telephone coils, was the iron in its virgin state to start with ? If the cores were magnetised strongly to begin with and then demagnetised, the properties in that state being used as the basis of design, it would always be possible to regain the initial condition without re-annealing.

The AUTHOR (communicated): In reply to Mr. Pollock, it still seems probable that by using a suitable system of demagnetisation the bad effects of accidental strong magnetisation can be entirely wiped out. Gumlich and

Rogowski ("Annalen d. Physik," p. 235, Vol. XXXIV. (2), 1911), who worked with low values of **H**, seemed well satisfied with the results they got by demagnetising with a special apparatus giving extremely smooth reduction of the magnetising field. In the experiments described in the Paper there was some suspicion of magnetic ageing at room temperature in the case of the annealed wires. With good stalloy sheet this effect would be improbable, but the material from which the wires are drawn may quite possibly have this bad property. Further investigation seems desirable.

A Demonstration of Experiments on the Thermionic Properties of Hot Filaments. By DR. F. LLOYD HOPWOOD.

SHOWN AT THE MEETING OF MAY 14TH, 1920.

THE experiments shown were some of those described by Dr. Hopwood in the "Philosophical Magazine," March, 1915, page 362, in which the glowing filament of a carbon lamp and glowing filaments of nichrome and platinum in air are made to move under the influence of positively and negatively charged rods brought into or withdrawn from their vicinity; the character of the effects observed being such as to give a qualitative indication of the thermionic emission from the filaments. In addition, he showed a type of tilted electroscope in which the gold leaf was replaced by a narrow loop of Wollaston wire. When a current is passed through the wire so as to make it glow, it forms an electroscope of different sensitivity for +ve and −ve charges.

DISCUSSION.

Mr. W. R. COOPER asked if the experiments had been tried with lamps lacquered with coloured lacquers. He had seen one covered with a red lacquer, the filament of which would move about following movements of the finger outside.

Dr. D. OWEN asked what was the explanation of the asymmetry of the electroscope. Did it depend on the direction of the current in the filament, or did it work as well with alternating current?

Dr. HOPWOOD said it was possible with lacquered lamps that one got an adventitious charge on the globe which reacted on the filament. He mentioned that during the war he knew of a number of lamps which in order to obscure them were dipped in "tantalum paint." This was so easily electrified when touched by dry hands that about 50 per cent. of the filaments broke.

In reply to Dr. Owen, the electroscope worked equally well with alternating current.

DISCUSSION ON EINSTEIN'S THEORY OF RELATIVITY.

HELD MARCH 26, 1920.

Prof. A. S. EDDINGTON said: The subject which we are to discuss arises out of a very famous experiment called the Michelson-Morley experiment, carried out in the year 1887. It was an attempt to detect and measure our motion through the luminiferous aether.

As the earth moves round the sun during the year, so its velocity through the aether must vary. The aether then is streaming past us, at least at some time during the year, and suppose for definiteness the current at the present moment is down this hall. Let a ray of light start from the back of the hall and travel a measured distance to a mirror up here, be reflected, and travel back to the starting point. I think you will easily see that it will take longer—be delayed on account of the current. Suppose at the same time we send another ray of light an equal measured distance across the hall and back. This also is delayed by the current, but it is easy to calculate that the delay is less in this case. We start the two rays together and let them have a race; the across-stream beam is the least delayed, and ought to arrive back first. Actually, when the experiment was tried with all possible refinements, the result was a dead-heat.

To make the conditions perfectly fair, the two courses, which were marked out in the apparatus used, could be interchanged by rotating the apparatus through a right angle. What had been the up-and-down arm became the transverse arm, but still the result was a dead-heat.

There is only one way out of this strange result, and it was suggested by FitzGerald. When we turn an arm of the apparatus, which was originally across stream, so that it is along stream, it must automatically contract by just the amount which compensates for the greater delay of the ray describing this course. And so, whichever arm of the apparatus is placed up stream, it immediately becomes the shorter.

We had been keeping an eye only on the effect of the current on the light waves; but the current may also have an effect on our apparatus. It appears that it has the effect of making a minute change in its dimensions. The change is the same whatever the material of the apparatus, and it is exactly of the amount needed to conceal the effect looked for.

When we consider the matter carefully, it is not so surprising after all. The size and shape of the material apparatus is maintained by the forces of cohesion, which are, presumably, of an electrical nature, and have their seat in the aether. It will not be a matter of indifference how the aether is streaming past, and there will be a readjustment of the molecules when the flow is altered.

But what does seem surprising is that the readjustment of size should just hide the effect we were hoping to find. It looks almost like a conspiracy. There have been three or four other experiments tried since then—of more technical kinds—all in the hope of detecting our motion through the aether; but they have all been defeated by the same kind of conspiracy. And to turn from experiment to theory, we find the same kind of conspiracy even in our mathematical equations. Perhaps you know how sometimes, just as you are hoping to get an important result, the equations evade you, and, instead of telling you what x is, they announce the solemn but irritating truth that $0=0$. It is somewhat like this in our electromagnetic theory: the velocity through the aether appears abundantly in the formulæ, but, whenever we try to run it down, it drops out, and refuses to be equal to anything in particular

In consequence of these conspiracies, a great generalisation has been put forward, known as the restricted Principle of Relativity. *It is impossible by any conceivable experiment to detect the velocity of a system through the aether.* That is to say, the conspiracy is a general one; and, in fact, it is in the nature of things that this motion is undetectable.

We make this generalisation and build a branch of science on it, in the same way as we make the generalisation that it is impossible to construct a perpetual-motion machine, and build the science of thermodynamics upon it. The case for the Principle of Relativity seems to me as strong as the case for the denial of the possibility of perpetual-motion machines.

We have seen that the concealment of our motion through the aether involves changes of length of moving material. I can best explain the nature of these changes by considering a rather extreme case. The statements I am going to make are all precise mathematical deductions from the principle just enunciated.

Suppose that an aviator flies past us at the rate of 161,000 miles a second. I shall assume that he is in a comfortable travelling conveyance, in which he can move and act normally, and that his length is in the direction of flight. If we could catch a glimpse of him as he passed, we should see a figure about 3 ft. high, but with the width and girth of a normal human being. And the strange thing is that he would be entirely unconscious of his undignified appearance. Looking in a mirror in the car he would see his usual figure. Moreover, if he looked down on us, it would appear to him that we were distorted and had undergone the flattening process.

If we watched the aviator closely, we should infer that he was unusually slow in his movements, and that all events in the car round him were similarly retarded, as though time had half-forgotten to go on. His cigar would last twice as long as one of ours. This slowing down remains after we have applied corrections for the time of transit of the light coming from him. (There is an additional apparent slowing due to the Doppler effect which has been allowed for.)

Here, again, reciprocity comes in, and the aviator perceives nothing unusual in his own surroundings, but thinks that we have gone sluggish. Our cigar lasts twice as long as his.

Which is right? Are we or the aviator? Each of us is using a different reckoning of space and time. To assert that we are right implies that space and time according to our reckoning are *the* space and time, and the aviator's space and time are mathematical fictions; but, owing to the complete reciprocity, this distinction corresponds to nothing experimental. We shall take the view that there is no absolute space and time in Nature, but only space and time reckonings for different observers, which are all on the same footing. The terms space and time only have a meaning relative to an observer, and differ according to his motion.

All this becomes easy to explain if the reality is four-dimensional, and the objects seen by us and the aviator are different three-dimensional appearances or sections of the same four-dimensional things. When a penny is viewed from different positions the observers see different two-dimensional appearances; but it would be absurd to maintain that one of these appearances was the real object, and that the other observers must make corrections on account of their positions. So when we see and measure an oblong block, and the aviator observing the same object finds it to be a cube, it is absurd to suggest that the oblong block is the real object, and that the aviator must correct for his motion. The cube and oblong are only different appearances of the one object, which is four-dimensional. It may be asked why do we not realise the four-dimensional world as vividly as we realise its three-dimensional section ? The reason is that we have two eyes, which are continually telling us that the world has to be looked at from more than one position, and the brain has responded and evolved the perception of solid relief which enables us to synthesise the appearances from different positions. But we are not similarly reminded that it has to be viewed from different motions. Perhaps if we were endowed with pairs of eyes in rapid motion relative to one another we should have evolved a faculty for combining appearances with different motions, and so perceive the relief in the fourth dimension.

In the world of four dimensions, length and duration are components of a single extension between two events. This extension is the same for everybody, being intrinsic in Nature. But observers, according to their motions, take different directions for resolving it into its two components length and duration.

The observer himself is a four-dimensional object; his shape is very elongated—practically linear. As we more usually say, he has considerable extension in time and more moderate dimensions in space. It is natural that the order of events which is parallel with his length should be distinguished by him from the remaining dimensions of the world. He distinguishes that order as time. Each observer takes for time the direction of his own extension—which, from the ordinary point of view, means his track of motion through the world.

If we view Nature in itself abstracting everything which is merely contributed by the observer, not only is space fused with time, but the more familiar attributes of both have vanished. Indeed, it is fair to say that space and time are contributed by the observer, and what is left in Nature is only a certain ordering of events, which is scarcely recognisable as space or time. There is no such thing as shape, no distinction of straight and crooked, no measure of an angle. We cannot describe this world in familiar language, because we have abstracted from it those relative aspects which might make it familiar.

Besides revising our conceptions of space and time, Einstein's theory gives altogether new ideas about force. It does not affect the forces of material contact—tension and thrust; but it affects the mysterious field of force—gravitation—which is a sort of latent influence round massive bodies ready to act if any particle enters the field.

The first thing which falls a victim to the new ideas is Newton's first law of motion—or, at any rate, the usual conception of its meaning. Every body moves uniformly in a straight line if undisturbed by forces. I suppose I am right in assuming that we think of this as implying that a body has an innate tendency called inertia, which makes it move uniformly in a straight line; but this can be overcome by active causes called forces. But I have said there is no distinction between straight and crooked in Nature; it is the observer who decides whether a certain locus shall be considered straight, and there is no such distinction of active and passive tendencies. All we can say is that there are certain natural tracks laid down in the space-time world—tracks which particles actually do follow unless some other material particles hit them. We call these natural tracks *geodesics*; they are laid down in Nature independently of an observer. Their courses must be determined by conditions in the universal substratum of things—the aether, the world, or whatever name you like to give it. For the present it seems useless to analyse them farther; we take these natural tracks and study their laws; this study is the geometry of the world.

What happens if the observer himself deviates from his natural track ? He can only do this if something is hitting him. He then perceives that he is in a field of force. The field is simply his own disturbance of path transferred to the objects round him, just as the diurnal revolution of the stars is our own rotation transferred to the stars. The observer gives different names to the field of force according to the conditions which produce his disturbance. If it is due to the earth whirling him round once a day, he says he is in a field of the earth's centrifugal force. If it is due to the molecules of the ground bombarding the soles of his boots, accelerating him upwards, he says he is in the field of the earth's gravitational force; but these fields are of essentially the same nature. This hypothesis has immediate consequences of great importance. We understand centrifugal force completely, and a little reflection will show that a ray of light is deflected by a field of centrifugal force, just as a material particle with the same velocity would be. (We recognise, of course, that this is a spurious effect due to our own rotation.) It follows at once that light can be deflected by gravitation, since the nature of the two fields is the same.

I suppose that my statement that the gravitation which we see manifested around us is simply due to the ground shoving against our feet requires a little more explanation. We generally agree that there is no absolute motion, and that all one can say is that two things are moving relatively to one another. So, when Newton saw the apple fall, all one can say is that there was a relative acceleration of the apple and Newton, without apportioning the responsibility to either. Still, there may be an easy and a paradoxical way of describing things; and since the apple was entirely free, suspended in space, whereas Newton was being bombarded upwards by the molecules of his chair, it seems the natural thing to attribute the acceleration to Newton. We are taking the geodesics as standard, and attribute accelerations relative to them to active causes—material impacts, about which there is nothing mysterious. To take the straight line as standard has two objections: (1) that it is a locus depending on the observer, without significance in Nature; (2) that the accelerations relative to it must be attributed to an invisible influence called the force of gravitation.

It is difficult to show without mathematics how Einstein proceeded from the consideration of these geodesics to deduce his new law of gravitation. The field of force round the sun can be specified completely if we state the courses in space and time of all possible particles projected in any manner in it. That is what Kepler did. We must find some way of unifying this cumbrous mass of knowledge. We cannot accept Newton's method of accomplishing this, because it introduces things relative to particular observers which we cannot believe have any *locus standi* in the true laws of the external world. Now, all the possible tracks together form a tangled web. A similar sort of web is formed by all the lines of shortest (or longest) length in a non-Euclidean space. If we can correlate such a non-Euclidean space, so that one web is an exact reproduction of the other, then we can obtain a concise summary of the whole mass of detail by merely stating the kind of geometry of the correlated space. Since there is some law governing the tracks, there must be some limitation on the correlated geometry. When we consider the possible ways of classifying non-Euclidean geometries, we find there are not many broad divisions; and there is only one reasonably likely law of gravitation, or limitation on the geometry, suggested. By proceeding in this way we confine attention to those things which have an intrinsic significance in Nature, and keep out things which appear only in knowledge relative to some observer, which are obviously unsuitable in formulating a general law of Nature.

The law thus suggested turns out to be almost equivalent to Newton's law in all ordinary applications; and in the exceptional cases where they differ appreciably—the deflection of light and the motion of perihelion of Mercury—it is the new law which is confirmed.

I have spoken of the non-Euclidean space as *correlated* with the field of gravitation. It turns out, however, that this space must actually be identical with the space and time of perception and of physical measurement—at any rate, far beyond possible experimental accuracy. It is in connection with this point that the third crucial test—the displacement of the Fraunhofer lines—comes in.

Probably this final identification of the field of force round the sun with the non-Euclidean properties of the physical space-time which we perceive there is the most difficult part of this theory. The essential point is that there is something going on in Nature whose absolute character we cannot attempt to define; we can only describe it by the effects to which it leads in our physical measures. Now, this absolute thing has effects of different kinds. Fields of force are one of these effects, and if we describe in terms of fields of force, we get a mechanical description of Nature, not very unlike Newton's description of gravitational force. The pulsations of clocks and the extensions of material scales measure other effects of the same thing, and if we describe in terms of these we get a geometrical description of Nature with non-Euclidean space and time. It is the strength of Einstein's theory that it brings out the unity underlying the two descriptions; that, since it is the same thing in Nature that is at the back of both, we can use the same description both for those aspects of Nature which are usually regarded as mechanical and those which are regarded as geometrical; and it is through this unification that the theory has achieved such great progress.

Mr. Bolton: Prof. Eddington mentioned that to an aviator flying at a high speed people on the ground would appear contracted in the direction of motion and *vice versa*, and appeared to deduce from this that certain dimensions of the interferometer in the Michelson-Morley Experiment would be contracted. He did not see the connection, as there was no relative motion between the observer and the instrument.

Prof. Eddington pointed out that the experiment gave a null result because of this fact; but the mathematician had deduced the contraction by imagining himself at rest relative to the æther, and therefore having a velocity relative to the instrument.

Sir George Lane Fox-Pitt asked if the observer was always regarded merely as a seeing person rather than a thinking person.

Prof. Eddington said that by "observer" he meant a person armed with his perceptions and with scientific apparatus where necessary to aid him in making measurements.

Dr. H. Levy said it was difficult to see why of all the possible tracks which a particle might follow it should always elect to follow a geodesic.

Prof. Eddington said one reason for identifying the natural tracks of particles with geodesics was that a particle must take a unique track definable without reference to an observer. The only track which can be defined uniquely in this way is a geodesic.

Mr. T. Smith asked if light in a gravitational field became to any extent polarised.

Prof. Eddington did not think there was any evidence of this.

Prof. A. O. Rankine said, apropos of Mr. Smith's question, that a few months ago in collaboration with Dr. Silberstein he had carried out some experiments on this point, the results of which were in the course of publication. Silberstein had shown that according to Einstein's theory a gravitational field should produce an effect on the velocity of light polarised in a plane parallel to the field. The same theory showed, however, that light plane-polarised at right angles to the field would be affected to the same extent, and it was desired to test how far this equivalence was in accordance with fact. A beam of light, plane polarised at 45° to the vertical, was projected in a horizontal direction and examined for ellipticity some distance away. It is evident that if the vertical and horizontal components travelled at different rates, a phase difference would be introduced which would result in elliptical polarisation. No trace of ellipticity was detected, the deduction from Einstein's theory being confirmed. Perhaps the most interesting fact emerging from the measurements was the extraordinary degree of precision with which the two velocities were found to be equal. The difference of optical path of the two components was certainly less than one three hundredth of a millimetre in the distance travelled in a second., viz., three hundred thousand kilometres, assuming a gravitational field equal to that of the earth to hold over the whole distance.

Dr. VINCENT asked if the theory of Relativity applied to the Kinetic Theory of Gases would not oblige us to alter our ideas on high temperatures. The conclusion was probably fallacious, but it appeared at first sight as if the electron could not move with a velocity greater than that of light, and that consequently there would be an upper as well as a lower limit to the absolute temperature scale.

Prof. EDDINGTON replied that with our usual definition of temperature as proportional to the energy of the atoms there was no upper limit; but if the scale were chosen to depend not on the energy but on the square of the velocity, it would reach a finite limit.

SIR JOSEPH LARMOR, in a communication read by the Secretary, said the charm of the Einstein train of ideas seems to be traceable to several causes. In the first place it has revived the interest in the Riemann differential geometry, itself a transcendental development of the ideas of the Gaussian theory of surfaces, and the admiration of Gauss for the train of thought of his disciple is on record. Then again, as it loosens all foundations, its ideas are capable of development in unexpected directions. How far it is from being a determinate theory of the universe will appear to anyone who dips into the writings of its developers. Indeed, if it became fixed and rigid and ceased to reveal unexpected vistas, much of the fascination would be gone. For example, the crowning consideration for the most general relativity, on which special stress is laid by Lorentz, and by Eddington, is the assertion of Einstein that it is only the intersections of the world-lines of particles that can constitute events, and so long as these points retain the same order along their lines, we may give body to their collocation by filling up with any kind of intermediate spacial scaffolding that we please. There is nothing to restrict it at all. Yet for all that we proceed straightway to assume a space with its ten gravitational potential functions, and work out its mathematics in the necessary elaboration; while we learn that ultimately there can really be nothing to limit our choice of such a framework. I observe that Langevin, being a Frenchman, does not shrink from blunt formulation of the full logical consequence. It is, therefore, possible, says he (Société française de Physique, Feb. 6th, 1920), to make enunciation of the principle of generalised relativity, in the assertion that the laws of physics have the same form for all systems of reference, in motion (or deformation) anyhow, in virtue of the introduction in each case of its own special field of gravitation suitably distributed. The ensuing problem is then to explain how it is that astronomers, since Newton's time, have persisted in one special and very precise illusion about the distribution of gravitation, whereas really an unlimited choice was open. One answer seems to involve swinging over to the opposite point of view, to the youthful dogmatism of Clifford, that if space could be fully known the whole material universe would be found to be involved implicitly within it. The general dialectic can thus develop many-sided and indefinite; the necessary coordinating consideration appears to be present in a somewhat latent postulate of one space-time continuum, which this unlimited multiplicity constitutes merely different modes of describing. If we would thus reach at bottom an underlying scheme of space and time after the manner of Newton, why should the natural historical order of development in terms of an æther be ruled out *a priori*?

The guiding philosopher for this type of theory is claimed by Einstein to be Hume, filtered through Mach; and it certainly affords a wider field for the play of his scepticism than his own illustrations.

The primary consideration in all sceptical analysis is to know where to stop, and, as with Kant in Hume's time, to replace further destructive criticism by constructive hypothesis. For the unresolvable essence of relativity appears to be that we cannot get on without assuming *some foundation* to which phenomena are referred, and with respect to which they are ordered to the degree that is necessary for our reasonings.

It has not escaped the notice of the creators of the analytical theory that the whole of it is expressible as a corollary, of novel type it is true, to the fundamental physical principle of least action, which adopts that principle to take in the phenomena of gravitation so as to place them in analytical relation to the other physical principles of nature. Of the fundamental character of that principle Helmholtz, following upon Hamilton and Kelvin and Maxwell, was not the least of the exponents. The original criteria of stress-energy-momentum tensors then shrink to their proper subordinate position. There is then nothing revolutionary about this brilliant theory, unless it be the fiction that time is merely an additional dimension of space, which is inconsistent with the very language in which it has to express itself, and at variance with the different conditions for physical determination in space and in time. That it is convenient to use mixed co-ordinates of time and space is another matter whose bearings were beautifully elucidated by the application long ago of the expressive phrase *local time* by Lozentz: it is local time varying irregularly from place to place that allows gravitation to come within the least action scheme.

Mr. A. W. **Ravenshear** said that in the Michelson-Morley experiment the contraction of the apparatus could not be measured by measuring rods or anything of that kind. It therefore required reference to some other system of measurement. What was this?

Prof. **Eddington** repeated that no contraction could be directly measured by this experiment. We are led to expect it on the assumption that the apparatus is moving relative to the æther. At any moment we do not know whether we are moving relative to the æther or not; but at some parts of the year we must be, so that if the results of the experiment are always negative we have to deduce the contraction.

A **Speaker** asked how, if curvatures in space only exists round matter, accommodation was found in the theory for the existence of free energy.

Prof. **Eddington** remarked that energy also gave curvature.

Mr. C. O. **Bartrum** pointed out that the Michelson-Morley experiment compared the velocity of light athwart the æther stream with the mean of the velocities up and down the stream. Had any experiment been made comparing the velocity up the stream with that down the stream, and did Prof. Eddington consider such an experiment was possible? He asked the question because he had seen it stated that the work of M. Q. Majorana had shown these velocities to be equal.

Prof. **Eddington** said he had heard that statement, but was not in agreement with it. Majorana's work was to show that there was no change in the velocity relative to the observer due to motion of the source or of a reflecting mirror. He did not think an experiment could be devised to test the point referred to as it was necessary to determine the times of arrival and departure at the two ends of the path. This could only be done by clocks, and if you use only one, you must correct for the velocity of light, which is the thing you are trying to measure.

Mr. T. **Smith** asked if any of the conclusions of the theory would be modified if we were assumed to be the dealing with group velocities rather than wave velocities. If he remembered rightly the most accurate determinations had revealed a difference even in vacuum.

Prof. **Eddington** was unable to say offhand.

Dr. E. H. **Rayner** asked what happened if the velocity of a body exceeded the velocity of light.

Prof. **Eddington** pointed out that a body would require infinite energy to attain this velocity, and so could not exceed it.

XXIV. *On Tracing Rays through an Optical System.* (*Third Paper*). *By* T. SMITH, *B.A.* (*From the National Physical Laboratory.*)

THE great majority of the rays that are traced by optical computers through a train of lenses lie in a plane containing the axis. The calculations for such rays are very simple compared with those for rays not remaining in the same plane during their passage through the system, but are distinctly more laborious than for those rays which lie close to the optical axis. The equation connecting conjugate axial points when refraction takes place at a single surface may be modified* in a variety of ways to meet the case of rays reaching the surface at large angles and at considerable distances from the axis. It has been pointed out that in some special cases these modified equations may provide a more rapid means of tracing rays than the usual trigonometrical methods. The object of the present note is to illustrate how such equations may be fitted into a scheme for ray tracing of quite general applicability.

The usual equation for the refraction of paraxial rays is

$$\frac{\mu'}{x'}-\frac{\mu}{x}=\frac{\mu'-\mu}{r}=(\mu'-\mu)R=\kappa,$$

or if m is the transverse magnification

$$x\kappa=-\mu(1-1/m),\quad x'\kappa=\mu'(1-m).$$

One of the generalised forms is

$$\frac{\mu'}{x'-p}-\frac{\mu}{x-p}=\frac{\mu'-\mu}{r-p}=j, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or if $\mathfrak{M}$ is the sine ratio $\mu\sin\psi/\mu'\sin\psi'$, where ψ and ψ' are the inclinations of the ray to the axis before and after refraction,

$$(x-p)j=-\mu(1-1/\mathfrak{M}),\quad (x'-p)j=\mu'(1-\mathfrak{M}) \quad . \quad . \quad (2)$$

Such forms may be justified immediately, for the sine ratio depends only on the relative distances of the object and image intersection points from the centre of curvature, and is, therefore, undisturbed by replacing the actual refraction by paraxial refraction at a concentric surface; moreover, by changing the radius of curvature the two conjugate points may be made conjugate for paraxial refraction. The satisfaction of these conditions evidently defines the refraction completely.

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* Trans. Opt. Soc., Vol. 20., p. 23.

It has been shown* previously that whenever refraction in an axial plane takes place without aberration, the perpendiculars to the ray from the vertex of the refracting surface before and after refraction are equal. If, then, refraction according to the paraxial law is to account satisfactorily for this non-paraxial ray, the vertex must be so chosen that these two perpendiculars are equal. This result also follows at once from equations (2) if they are multiplied by sin ψ and sin ψ' respectively. If, as before, the perpendiculars from the actual vertex are denoted by $\mathfrak{h}$ and $\mathfrak{h}'$,

$$\mathfrak{h}+p\sin\psi=\mathfrak{h}'+p\sin\psi', \quad . \quad . \quad . \quad . \quad (3)$$

and since the refraction follows paraxial laws exactly

$$\mu'\sin\psi'-\mu\sin\psi=-\mathfrak{j}(\mathfrak{h}+p\sin\psi). \quad . \quad . \quad . \quad (4)$$

Equations (3) and (4) are equivalent to (2), but are in a more desirable form for ray tracing since $\mathfrak{h}$ and $\mathfrak{h}'$ are always finite, while x and x' may attain any real values.

If the angular aperture at which refraction takes place is -2α, the deviation is 2δ, and the angle made with the axis by the join of the intersection of the rays with the aplanatic surfaces is 2γ, the inclinations ψ, ψ' to the axis and the angles φ, φ' with the normal are given by

$$\psi=\gamma-\alpha-\delta,\ \psi'=\gamma-\alpha+\delta,\ \varphi=\gamma+\alpha-\delta,\ \varphi'=\gamma+\alpha+\delta. \quad . \quad (5)$$

Since the equivalent vertex is the point where the external bisector of the angle made by the rays meets the axis, it is easily seen that

$$\frac{r-p}{r}=\frac{\cos(\gamma+\alpha)}{\cos(\gamma-\alpha)},$$

or

$$p=\frac{2er}{1+e}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$e=\tan\alpha\tan\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This is the equation given previously for finding p. Evidently p will ordinarily be a very small quantity. In certain cases its value may be written down with sufficient accuracy without calculation, but in general it must be determined from the known values of $\mathfrak{h}$ and sin ψ. The equation expressing it in terms of these variables is a bi-quadratic. This would be expected, since there will be solutions corresponding to both the internal and the external bisectors of the angle between the rays, and two further solutions corresponding to refraction at a surface of opposite curvature, which may evidently satisfy the conditions. The high order of this equation occasions no difficulty, as the small root required is given very approximately by the two lowest terms of the equation, the remaining terms of which may be regarded as corrections to this approximate value. The value of p so found may be applied to (4) to find $\sin\psi'$, and the value of $\mathfrak{h}$ for the next surface may be determined by applying the usual equation of transference, reckoning the separation from the equivalent vertex.

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* Proc. Phys. Soc., Vol. XXX., p. 227.

The object of this method of calculation is to avoid the use of the equation $\psi'=\psi-\varphi+\varphi'$ applied to the angles themselves, this, the essential characteristic of the trigonometrical method, necessarily involving four references to tables, and usually an interpolation for each. The value of any alternative procedure must lie in its furnishing the value of $\sin\psi'$ to a corresponding degree of accuracy in less time.

From the known values of $\sin\psi$, $\mathfrak{h}$ and R, the sines of the angles of incidence and emergence are found from the usual formula

$$\mu'\sin\varphi'=\mu\sin\varphi=\mu(\mathfrak{h}R+\sin\psi). \quad . \quad . \quad . \quad (8)$$

It is then evident from equations (5) that to a first approximation

$$2\alpha=\sin\varphi-\sin\psi, \quad 2\gamma=\sin\varphi'+\sin\psi,$$

and therefore

$$4e=\mathfrak{h}R(\sin\varphi'+\sin\psi),\ 2p=\mathfrak{h}(\sin\varphi'+\sin\psi). \quad . \quad (9)$$

This is an underestimate for p, the exact relation being

$$2p=\frac{\mathfrak{h}(\sin\varphi'+\sin\psi)}{\cos(\gamma-\delta)\cos(\gamma-\alpha)\cos(\alpha+\delta)},$$

which suggests as a corrected form

$$2p=\frac{\mathfrak{h}(\sin\varphi'+\sin\psi)}{1-\frac{1}{8}\{(\sin\varphi+\sin\psi)^2+(\sin\varphi'-\sin\psi)^2+(\sin\varphi'-\sin\varphi+2\sin\psi)^2\}} \quad (10)$$

and this is sufficiently accurate for all ordinary purposes. Other forms may be found without difficulty by constructing equations satisfied by e. Thus from equations (5)

$$\sin\varphi-\sin\psi=2\sin\alpha\cos(\gamma-\delta),\ \sin\varphi+\sin\psi=2\cos\alpha\sin(\gamma-\delta)$$
$$\sin\varphi'+\sin\psi=2\sin\gamma\cos(\alpha+\delta),\ \sin\varphi'-\sin\psi=2\cos\gamma\sin(\alpha+\delta).$$

Combining these equations with e

$$\sin\varphi-\sin\psi+e(\sin\varphi+\sin\psi)=2\sin\alpha\cos\delta/\cos\gamma,$$
$$\sin\varphi'+\sin\psi+e(\sin\varphi'-\sin\psi)=2\sin\gamma\cos\delta/\cos\alpha,$$
$$\sin\varphi-\sin\psi-e(\sin\varphi'-\sin\psi)=2\tan\alpha\cos(\alpha+\gamma)\cos\delta,$$
$$\sin\varphi'+\sin\psi-e(\sin\varphi+\sin\psi)=2\tan\gamma\cos(\alpha+\gamma)\cos\delta,$$
$$(\sin\varphi-\sin\psi)(\sin\varphi'+\sin\psi)-e(\sin\varphi+\sin\psi)(\sin\varphi'-\sin\psi)$$
$$=4\sin\alpha\sin\gamma\cos(\alpha+\gamma),$$
$$(\sin\varphi-\sin\psi)(\sin\varphi'+\sin\psi)-e^2(\sin\varphi+\sin\psi)(\sin\varphi'-\sin\psi)$$
$$=4e\cos(\alpha+\gamma)\cos\delta\cos\psi,$$

from which $\cos(\alpha+\gamma)$ and $\cos\delta$ have to be eliminated. Thus

$$4e[\sin\varphi-\sin\psi-e(\sin\varphi'-\sin\psi)]\,[\sin\varphi'+\sin\psi-e(\sin\varphi+\sin\psi)]$$
$$=(1-e)[\sin\varphi-\sin\psi+e(\sin\varphi+\sin\psi)][\sin\varphi'+\sin\psi+e(\sin\varphi'-\sin\psi)]$$
$$\times[(\sin\varphi-\sin\psi)(\sin\varphi'+\sin\psi)-e(\sin\varphi+\sin\psi)(\sin\varphi'-\sin\psi)]$$
$$=[(\sin\varphi-\sin\psi)(\sin\varphi'+\sin\psi)-e^2(\sin\varphi+\sin\psi)(\sin\varphi'-\sin\psi)]^2\sec^2\psi.$$

The second of these equations is ordinarily to be preferred. Let the correcting factor $\mathcal{E}$ be taken as independent variable, where

$$4e\mathcal{E}=(\sin\varphi-\sin\psi)(\sin\varphi'+\sin\psi)\,. \quad . \quad . \quad . \quad (11)$$

The second equation then becomes

$$\mathcal{E}[\mathcal{E}^2-\tfrac{1}{4}\mathcal{E}(\sin^2\varphi+\sin^2\varphi'-2\sin^2\psi)$$
$$+\tfrac{1}{16}(\sin^2\varphi-\sin^2\psi)(\sin^2\varphi'-\sin^2\psi)]\cos^2\psi$$
$$=[\mathcal{E}^2-\tfrac{1}{16}(\sin^2\varphi-\sin^2\psi)(\sin^2\varphi'-\sin^2\psi)]^2 \quad . \quad . \quad . \quad (12)$$

with the solution

$$\mathcal{E}=1-\tfrac{1}{4}(\sin^2\varphi+\sin^2\varphi'+2\sin^2\psi)-\tfrac{1}{16}(\sin^4\varphi+\sin^4\varphi'+\sin^4\psi$$
$$-\sin^2\varphi\sin^2\varphi'-\sin^2\varphi\sin^2\psi-\sin^2\varphi'\sin^2\psi)-\ .\ .\ .\ .$$

The terms of the fourth order are obviously small in many cases, and since $\mathcal{E}$ is a correcting factor to a small quantity, they may often be neglected. The value of p is to be obtained from

$$2p=\frac{\mathfrak{h}\,(\sin\varphi'+\sin\psi)}{\mathcal{E}+\frac{1}{4}\,\mathfrak{h}\,R(\sin\varphi'+\sin\psi)}, \quad . \quad . \quad . \quad (13)$$

which agrees with (10) when terms of the fourth order in $\mathcal{E}$ are neglected.

One of the advantages to be expected from tracing rays by calculating the p's of successive surfaces is that under similar conditions of refraction the values of p are closely proportional to the thicknesses of the lenses if these are determined by the lens aperture, and are independent of the focal length. It will thus often happen that a computer who employs this system will be able from his past experience to assume sufficiently accurate values for p by basing them on the lens thicknesses. Nevertheless it is at times convenient to disregard p entirely, and to eliminate it from equations (3) and (4). It will be shown that this leads to a correcting factor quite symmetrical in φ, φ' and ψ.

Since there is no aberration when $\mathfrak{h}=0$, it is permissible to assume for p the form

$$p=\frac{\mathfrak{h}P}{1-P\sin\psi}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

which gives

$$\mathfrak{h}+p\sin\psi=\frac{\mathfrak{h}}{1-P\sin\psi}$$

and

$$\frac{r-p}{r}=\frac{1-P\sin\varphi}{1-P\sin\psi}$$

giving for (4)

$$\mu'\sin\psi'-\mu\sin\psi=-\kappa\mathfrak{h}/(1-P\sin\varphi). \quad . \quad . \quad . \qquad (15)$$

In this form the equation is not inconvenient for use, but an alternative form free from refractive indices

$$\sin\psi'-\sin\psi=(\sin\varphi'-\sin\varphi)\frac{1-P\sin\psi}{1-P\sin\varphi} \quad . \quad . \qquad (16)$$

is obtained by adding $(\mu-\mu')\sin\psi$ to both sides. By substituting in (3) this value of $\sin\psi'$ and by the original assumption the two equations

$$\frac{\mathfrak{h}'}{\mathfrak{h}}=\frac{1-P\sin\psi'}{1-P\sin\psi}=\frac{1-P\sin\varphi'}{1-P\sin\varphi} \quad . \quad . \quad . \qquad (17)$$

for $\mathfrak{h}'$ are found. It remains to find a formula for P. Since P vanishes with $\sin\varphi'+\sin\psi$, put

$$2PE=\sin\varphi'+\sin\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Comparing (13) and (14)

$$E=\mathcal{E}+\tfrac{1}{4}(\sin\varphi+\sin\psi)(\sin\varphi'+\sin\psi) \quad . \quad . \quad . \quad . \qquad (19)$$

and the elimination of $\mathcal{E}$ between (12) and (19) gives

$$E^4-E^3(1+s_2)+\tfrac{1}{4}E^2(s_2{}^2+s_1s_3+s_1{}^2+s_2)-\tfrac{1}{8}E(s_1+s_3)(s_1s_2-s_3)$$
$$+\tfrac{1}{64}(s_1s_2-s_3)^2=0 \qquad (20)$$

where

$$s_1=\sin\varphi+\sin\varphi'+\sin\psi, \quad s_2=\sin\varphi\sin\varphi'+\sin\varphi\sin\psi+\sin\varphi'\sin\psi,$$
$$s_3=\sin\varphi\sin\varphi'\sin\psi,$$

showing that E is symmetrical in φ, φ', ψ. This is also seen by noting that

$$E=\cos\alpha\cos\delta\cos(\alpha+\delta)$$
$$=\tfrac{1}{4}\{1+\cos(\varphi-\psi)+\cos(\varphi'-\psi)+\cos(\varphi'-\varphi)\} \quad . \quad (21)$$

From either (20) or (21) it is easy to derive the solution

$$\left.\begin{aligned}E&=1+\tfrac{1}{8}(S_1{}^2-3S_2)+\tfrac{1}{32}(S_2{}^2-3S_4)+\tfrac{1}{64}(S_2S_4-3S_6)+\ldots\\&=\tfrac{1}{8}[-1+S_1{}^2+(3-\tfrac{1}{2}S_2-\tfrac{1}{8}S_4-\tfrac{1}{16}S_6\ \ldots\ )^2]\end{aligned}\right\}(22)$$

where

$$S_n=\sin^n\varphi+\sin^n\varphi'+\sin^n\psi,$$

an exact solution when required being formed from the sum of the three cosines. In numerical calculations equations (16) and (17) will be used in the forms

$$\sin\psi'-\sin\psi=(\sin\varphi'-\sin\varphi)\left[1+\frac{(\sin\varphi-\sin\psi)(\sin\varphi'+\sin\psi)}{2E-\sin\varphi(\sin\varphi'+\sin\psi)}\right]$$

$$\mathfrak{h}'=\mathfrak{h}\left[1+\frac{(\sin\varphi-\sin\varphi')(\sin\varphi'+\sin\psi)}{2E-\sin\varphi(\sin\varphi'+\sin\psi)}\right],$$

the denominator being frequently represented with sufficient accuracy by

$$2-\tfrac{1}{4}[(\sin\varphi+\sin\varphi')^2+(\sin\varphi+\sin\psi)^2+(\sin\varphi'-\sin\psi)^2]$$

where the sines in each term are connected by the opposite sign to the one in the terms of the numerators. It should be noted that the correcting term in the first of these equations is obtained by multiplying that for the second equation by $-R$, as is evident from the initial equation for $\mathfrak{h}'$ in terms of φ' and ψ'.

When the cosines of the angles are to be found, the denominator will be most simply calculated from the formula

$$\tfrac{1}{4}\{(\sin\psi-\sin\varphi+\sin\varphi')^2+(\cos\psi+\cos\varphi+\cos\varphi')^2-1\}$$

and it is in this form that the equations have been found most convenient in practice. It will be noted that since the difference between any cosine and unity enters into the equations as a correction to the aberration alone and does not affect the normal refraction according to Gaussian laws, a knowledge of the cosine to a comparatively small number of figures yields an accuracy corresponding to the use of a greater number of figures in the usual trigonometrical calculation. It thus becomes possible to obtain a very high degree of accuracy by constructing a short four-figure table giving the cosines of angles in terms of their sines, such as that given below. The interpolations involved in the use of this table only affect the last figure and can always be made mentally. They thus differ fundamentally from those required with the tables ordinarily used, and the time usually spent in turning over the pages of tables to reach the angles required is entirely avoided. Calculations made by this method on photographic lenses of complex form with angles of incidence reaching 60° show that the final results are accurate to about 3″ in direction, with a corresponding degree of accuracy in the position of the ray. Unlike the methods described previously the perpendicular after refraction is obtained before the new direction of the ray is found. The order of operations is indicated by the equations

$$\begin{gathered}
\sin\varphi_\lambda=\sin\psi_{\lambda-1}+\mathfrak{h}_\lambda R_\lambda \\
\mu_\lambda\sin\varphi'_\lambda=\mu_{\lambda-1}\sin\varphi_\lambda \\
\mathfrak{h}'_\lambda=\mathfrak{h}_\lambda+ \\
\frac{4\,\mathfrak{h}_\lambda(\sin\varphi_\lambda-\sin\varphi'_\lambda)(\sin\varphi'_\lambda+\sin\psi_{\lambda-1})}{(\sin\psi_{\lambda-1}-\sin\varphi_\lambda+\sin\varphi'_\lambda)^2+(\cos\psi_{\lambda-1}+\cos\varphi_\lambda+\cos\varphi'_\lambda)^2-1} \qquad (23) \\
\sin\psi_\lambda=\sin\varphi'_\lambda-\mathfrak{h}'_\lambda R_\lambda \\
\mathfrak{h}_{\lambda+1}=\mathfrak{h}'_\lambda+t_\lambda\sin\psi_\lambda .
\end{gathered}$$

The formulæ as they stand are applicable to every case that can arise, and it does not appear desirable to depart from them in any case with the possible exception of a flat surface, where the value of $\sin\psi_\lambda$ is obtained from that of $\sin\psi_{\lambda-1}$ by the law of refraction, and $\mathfrak{h}'_\lambda$ may be obtained from the relation

$$\frac{\mathfrak{h}'_\lambda}{\cos\psi_\lambda}=\frac{\mathfrak{h}_\lambda}{\cos\psi_{\lambda-1}}.$$

If this equation is used the cosines must be calculated directly, the table being insufficiently accurate. If the table is to be utilised the equation should be expressed in the form

$$\mathfrak{h}'_\lambda=\mathfrak{h}_\lambda\left[1+\frac{(\sin\psi_{\lambda-1}-\sin\psi_\lambda)(\sin\psi_{\lambda-1}+\sin\psi_\lambda)}{\cos\psi_{\lambda-1}(\cos\psi_{\lambda-1}+\cos\psi_\lambda)}\right].$$

Perhaps the simplest method of deriving the new equation for $\mathfrak{h}'$ from those given previously is to start from the relations

$$\frac{\mathfrak{h}'}{\mathfrak{h}}=\frac{\sin\varphi'-\sin\psi'}{\sin\varphi-\sin\psi}=\frac{\cos\varphi'+\cos\psi'}{\cos\varphi+\cos\psi},$$

COSINES IN TERMS OF SINES.

	0	1	2	3	4	5	6	7	8	9	*D*	D^{-1}
·70	·7141	·7132	·7122	·7112	·7102	·7092	·7082	·7072	·7062	·7052	−10	−10
1	·7042	·7032	·7022	·7012	·7001	·6991	·6981	·6971	·6960	·6950	−10	−10
2	·6940	·6929	·6919	·6908	·6898	·6887	·6877	·6866	·6856	·6845	−11	− 9
3	·6834	·6824	·6813	·6802	·6791	·6781	·6770	·6759	·6748	·6737	−11	− 9
4	·6726	·6715	·6704	·6693	·6682	·6671	·6659	·6648	·6637	·6626	−11	− 9
5	·6614	·6603	·6592	·6580	·6569	·6557	·6546	·6534	·6523	·6511	−11	− 9
6	·6499	·6488	·6476	·6464	·6452	·6440	·6428	·6416	·6404	·6392	−12	− 9
7	·6380	·6368	·6356	·6344	·6332	·6320	·6307	·6295	·6283	·6270	−12	− 8
8	·6258	·6245	·6233	·6220	·6208	·6195	·6182	·6170	·6157	·6144	−13	− 8
9	·6131	·6118	·6105	·6092	·6079	·6066	·6053	·6040	·6027	·6013	−13	− 8
·80	·6000	·5987	·5973	·5960	·5946	·5933	·5919	·5906	·5892	·5878	−14	− 7
1	·5864	·5851	·5837	·5823	·5809	·5795	·5781	·5766	·5752	·5738	−14	− 7
2	·5724	·5709	·5695	·5680	·5666	·5651	·5637	·5622	·5607	·5592	−15	− 7
3	·5578	·5563	·5548	·5533	·5518	·5502	·5487	·5472	·5457	·5441	−15	− 7
4	·5426	·5410	·5395	·5379	·5363	·5348	·5332	·5316	·5300	·5284	−16	− 6
5	·5268	·5252	·5235	·5219	·5203	·5186	·5170	·5153	·5136	·5120	−16	− 6
6	·5103	·5086	·5069	·5052	·5035	·5018	·5000	·4983	·4966	·4948	−17	− 6
7	·4931	·4913	·4895	·4877	·4859	·4841	·4823	·4805	·4787	·4768	−18	− 6
8	·4750	·4731	·4712	·4694	·4675	·4656	·4637	·4618	·4598	·4579	−19	− 5
9	·4560	·4540	·4520	·4501	·4481	·4461	·4441	·4420	·4400	·4379	−20	− 5
·90	·4359	·4338	·4317	·4296	·4275	·4254	·4233	·4211	·4190	·4168	−21	− 5
1	·4146	·4124	·4102	·4080	·4057	·4035	·4012	·3989	·3966	·3943	−23	− 4
2	·3919	·3896	·3872	·3848	·3824	·3800	·3775	·3751	·3726	·3701	−24	− 4
3	·3676	·3650	·3625	·3599	·3573	·3546	·3520	·3493	·3466	·3439	−26	− 4
4	·3412	·3384	·3356	·3328	·3299	·3271	·3242	·3212	·3183	·3153	−29	− 3
5	·3122	·3092	·3061	·3030	·2998	·2966	·2934	·2901	·2868	·2834	−32	− 3
6	·2800	·2765	·2730	·2695	·2659	·2622	·2585	·2548	·2510	·2471	−37	− 3
7	·2431	·2391	·2350	·2308	·2265	·2222	·2178	·2132	·2086	·2039	−44	− 2
8	·1990	·1940	·1889	·1836	·1782	·1726	·1667	·1607	·1545	·1479	−53	− 2
9	·1411	·1339	·1262	·1181	·1094	·0999	·0894	·0774	·0632	·0447	...	...
·990	·1411	·1404	·1397	·1389	·1382	·1375	·1368	·1361	·1353	·1346	...	...
1	·1339	·1331	·1324	·1316	·1309	·1301	·1293	·1286	·1278	·1270	...	...
2	·1262	·1254	·1247	·1239	·1231	·1222	·1214	·1206	·1198	·1190	...	...
3	·1181	·1173	·1164	·1156	·1147	·1138	·1130	·1121	·1112	·1103	...	...
4	·1094	·1085	·1075	·1066	·1057	·1047	·1038	·1028	·1018	·1009	...	...
5	·0999	·0989	·0979	·0968	·0958	·0948	·0937	·0926	·0916	·0905	...	...
6	·0894	·0882	·0871	·0859	·0848	·0836	·0824	·0812	·0799	·0787	...	...
7	·0774	·0761	·0748	·0734	·0721	·0707	·0692	·0678	·0663	·0648	...	...
8	·0632	·0616	·0600	·0583	·0565	·0548	·0529	·0510	·0490	·0469	...	...
9	·0447	·0424	·0400	·0374	·0346	·0316	·0283	·0245	·0200	·0141	...	...

[*To face p. 258.*

whence

$$\frac{\mathfrak{h}'}{\mathfrak{h}}=\frac{(\cos\varphi'+\cos\psi')(\cos\varphi+\cos\varphi')+(\sin\varphi'-\sin\psi')(\sin\varphi-\sin\varphi')}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=\frac{1+\cos(\varphi'-\varphi)+\cos(\varphi'-\psi')+\cos(\varphi+\psi')-2\sin^2\varphi'}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=\frac{1+\cos(\varphi'-\varphi)+\cos(\varphi-\psi)+\cos(\varphi'+\psi)-2\sin^2\varphi'}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=\frac{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi+2\sin\varphi'+\sin\psi)(\sin\varphi-\sin\varphi')}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=1+\frac{2(\sin\varphi-\sin\varphi')(\sin\varphi'+\sin\psi)}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=1+\frac{2(\sin\varphi-\sin\varphi')(\sin\varphi'+\sin\psi)}{(\cos\varphi+\cos\psi)(\cos\varphi'+\cos\psi)-(\sin\varphi-\sin\psi)(\sin\varphi'+\sin\psi)}$$

two alternative forms for the denominator, which are suitable for use with the tabulated cosines, and may easily be resolved into the form already given in equation (23).

The cosine tables are available for all the operations which have been described in previous papers, provided the equations are arranged so that the cosines appear only in a term which may be regarded as a correction to the paraxial result. Such equations can always be found in a form convenient for calculation. The positions of the focal lines, for example, may be determined in this way. When it is desired to find errors of phase in the emergent wave front the equation given previously

$$\Sigma\frac{(\mathfrak{h}'-\mathfrak{h})\mu\sin\varphi}{\{(1+\cos\varphi)(1+\cos\varphi')(1+\cos\psi)(1+\cos\psi')\}^{\frac{1}{2}}},$$

is probably as suitable as any other form, and fits very conveniently into the present system of calculation.

The author desires, in conclusion, to thank Miss Gillman for checking the table of cosines, and both Miss Everett and Miss Gillman for comparative calculations of rays in various optical systems which have indicated the degree of accuracy obtainable as recorded in the paper.

ABSTRACT

Equations for tracing rays in an axial plane through an optical system have the normal refraction terms separated from those representing aberrations. By expressing the latter as a fraction with the

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$$=\frac{1+\cos(\varphi'-\varphi)+\cos(\varphi'-\psi')+\cos(\varphi+\psi')-2\sin^2\varphi'}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=\frac{1+\cos(\varphi'-\varphi)+\cos(\varphi-\psi)+\cos(\varphi'+\psi)-2\sin^2\varphi'}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=\frac{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi+2\sin\varphi'+\sin\psi)(\sin\varphi-\sin\varphi')}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

$$=1+\frac{2(\sin\varphi-\sin\varphi')(\sin\varphi'+\sin\psi)}{(\cos\varphi+\cos\psi)(\cos\varphi+\cos\varphi')+(\sin\varphi-\sin\psi)(\sin\varphi-\sin\varphi')}$$

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ABSTRACT

Equations for tracing rays in an axial plane through an optical system have the normal refraction terms separated from those representing aberrations. By expressing the latter as a fraction with the

first order aberration as the numerator and a correcting factor, which may take various forms, as the denominator, rays may be traced exactly through the system using a short table of cosines in terms of sines in place of the extensive tables, giving sines in terms of angles generally employed. A considerable saving of time is effected in the calculations and the estimation, without calculation, of the aberrations of other rays is facilitated.

DISCUSSION.

Mr. J. W. French communicated the following note: I have read with much interest Mr. Smith's very valuable contribution to the subject of optical computation. The principal advantage claimed for the system over, say, the Steinheil and Voit trigonometrical system, is, I understand, one of speed, as Mr. Smith's system does not involve the conversion of functions of angles to and from degrees and minutes. In order to test this point, I have had a cemented objective computed by both methods, the same operator and machine being engaged throughout. It would appear that the number of operations under Mr. Smith's system is much greater than in the case of the Steinheil system. Regarding each item that is recorded on paper by the computer when using a machine as indicating an operation, then for the complete objective the number of operations involved in Mr. Smith's method is 69, as compared with 33 in the Steinheil method. By the use of a machine having an integrating slide, it is possible that the number of paper entries involved in Mr. Smith's method could be substantially reduced, but the number of actual calculating operations would still appear to be considerably greater than in the Steinheil method. Thus, the number of additions or subtractions per surface on the Smith system is eight, as compared with four in the Steinheil system; the number of multiplications or divisions on the Smith system is eight, as compared with five on the Steinheil; the number of table references on the Smith system is three, as compared with three on the Steinheil. Now, with regard to addition, subtraction, multiplication and division, it is possible that Mr. Smith adopts short cuts which are not indicated in the Paper. If this is not the case, it would appear that the Steinheil system is nearly twice as quick as the Smith system, so far as these operations are concerned. With regard to table references, it is stated that in the trigonometrical method four references to tables are involved. I suggest that really only three table references are involved. It is true there are four angles, but in the case of an objective, at all surfaces after the first, one is known. When the object is at infinity there are only three angles involved, even at the first surface, since one of the four is zero. When the object is at a finite distance there are four angles involved at the first surface, and only three at succeeding surfaces. So far as table reference is concerned, it would appear that the only advantage that remains is that Mr. Smith, by an ingenious device, is able to obtain accurate results from four-figure tables, whereas tables of six or seven figures are required in the Steinheil method. It may be of interest to note that it is our general practice to use radian tables. In view of the small number of table references and the great difference between the two methods so far as time occupied in the other operations is concerned, and considering the facility which a trained computer acquires in the use of tables, I suggest that the Steinheil system is more rapid, unless, as already stated, Mr. Smith may employe a number of short cuts that have not been indicated. So far as accuracy is concerned, the proposed method appears to be all that Mr. Smith claims. In computing an ordinary telescope objective, the final angles obtained by the two methods corresponded to one unit in the sixth significant figure and the intercepts to three in the fifth figure.

Commander T. Y. Baker (communicated): The computing opticians' methods of ray tracing, although varying in detail as to conventions of signs, have all been practically of the same type—namely, successive calculations of the refractive equation $\mu \sin \varphi = \mu' \sin \varphi'$, and successive transference from one surface to the next usually by formulæ connecting the crossing point of the ray and the aperture of the point of its inter-section with the next surface. Such formulæ have all been purely trigonometrical, and, as Mr. Smith states, the calculation $\psi' = \psi - \varphi + \varphi'$ crops up for each surface.

Optical computation by such methods is slow. Probably, at the best, one man working alone cannot exceed a speed of 12 surfaces per hour, even with the use of a calculating machine to replace logarithmic computation. The slowness is mainly caused by the constant necessity of referring to tables of sines and the need for interpolation between tabulated values. A small improvement can be made if sines tabulated to seconds are available, although the greater bulkiness of the tables makes in itself for delay. It must, therefore, be a great pleasure to the calculating optician to find a scheme put forward, such as the present one, in which the troublesome parts of the ordinary methods are circumvented. In this scheme of work, the actual values of the angles of incidence, refraction or of inclination with the axis, never need to be found; their sines only are dealt with except in one place, and the whole set of formulæ could be expressed in an algebraic form without the use of a single trigonometrical function. Except for the tabulated values of $\sqrt{1-x^2}$, which is really the meaning of Mr. Smith's table, nothing is needed except ordinary logarithms, and not even those when the multiplications and divisions are done by calculating machine. I have had the opportunity, through Mr. Smith's kindness, of testing the speed of his scheme against that of ordinary methods, and, although nothing like full proficiency has yet been attained, we have, nevertheless, already reached the " five minutes per surface " mark, and it would appear as if the ultimate limit would be from 20 to 25 surfaces per hour. The slowest part of the scheme is the solution of equation 23, in which $\mathfrak{h}'$ is determined from $\mathfrak{h}$; in fact, this equation takes nearly half the time. Possibly on a different type of calculating machine, where numbers could be left on the machine for subsequent use, one might be able to economise time; but, with the machine that I was using, I found that equation 23 as it stands is not in the most suitable form, and prefer to use the following, which is readily established by geometry, and is not approximate in any form :—

$$\mathfrak{h}' = \mathfrak{h}\left[1 - 2 \sin \tfrac{1}{2}(\varphi' + \psi) \sin \tfrac{1}{2}(\varphi' - \varphi) \sec \tfrac{1}{2}(\varphi + \psi)\right].$$

The calculation of the expression

$$2 \sin \tfrac{1}{2}(\varphi' + \psi) \sin \tfrac{1}{2}(\varphi' - \varphi) \sec \tfrac{1}{2}(\varphi + \psi),$$

which is really of aberrational magnitude, can be made with sufficient accuracy by a specially constructed slide rule based on the following principles: If on a circle of unit radius three points be marked off on the circumference from a common zero, and subtending the three angles φ, φ' and $-\psi$ at the centre, the fraction above can be written

$$\frac{\text{Chord } (\varphi' \text{ to } -\psi) \times \text{chord } (\varphi' \text{ to } \varphi)}{2 \times \text{perpendicular from centre on chord } (\varphi \text{ to } -\psi)}$$

It remained, therefore, from the known values of the sines of the three angles to determine the chords and the perpendicular. This was easily done by constructing a scale of $\sin^{-1}x$. The actual distance, consequently, in this scale between the marked values of $\sin \varphi'$ and $\sin \varphi$ is the angle $\varphi' - \varphi$, and the chord corresponding to this angle can be read on the slide. A second slide read the length of the perpendicular. The accuracy obtainable depends, of course, on the range to which the values are marked and on the scale of the slide rule. Probably, for separated lens systems, 90 per cent. of surfaces will be such that sin φ, sin φ' or sin ψ are all below 0·3, and if the slide is made to go up to that value, the occasional cases outside the range can be dealt with by Mr. Smith's formula or directly by trigonometrical methods. The rough rule to guide one respecting the accuracy seemed to be that if the value of the above fraction had two zeros after the decimal point, it could be trusted to be right to about one in the fifth place. If there was only one zero, it was usually desirable to make the calculation independently of the slide rule. The above accuracy can, of course, be improved by increasing the scale of the slide rule, but it appears to be sufficient for separated lens systems. I have not had sufficient time to investigate to what extent the method is applicable to photographic lenses where the curves are steep and the apertures wide. In conclusion, I feel that Mr. Smith is to be congratulated on having evolved a system of ray tracing which looks to be invaluable for the optical computer, and I trust that steps will be taken to bring it prominently to the notice of instrument making firms. I have in mind several instances of delay in getting out new designs owing to congestion in the computation section, and a scheme that will enable a computer to double his output ought not to be lost sight of.

Mr. F. J. W. **Whipple** (communicated): It may be of interest to place on record a variant of Mr. Smith's formula (23), which appears to have advantages from the computer's point of view as well as the merit of simple geometrical proof. If P is the point where the ray cuts the refracting surface of which A is the vertex and C the centre, let AN, perpendicular to the incident ray, cut the bisector of the angle PCA at Q. $\mathfrak{h}$ is the sum of the projections of the equal lines PQ and QA on any perpendicular to the incident ray, so that

$$\mathfrak{h} = PQ[1+\cos(\psi+\varphi)]$$

Similarly $\mathfrak{h}' = PQ[\cos(\varphi-\varphi')+\cos(\psi+\varphi')]$

Hence
$$\frac{\mathfrak{h}'}{\mathfrak{h}} = \frac{\cos(\varphi-\varphi')+\cos(\psi+\varphi')}{1+\cos(\psi+\varphi)}$$

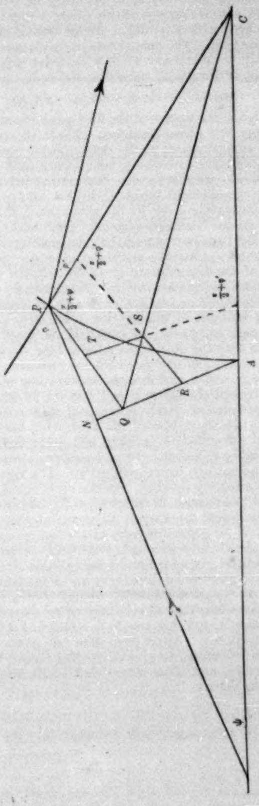
$$=1-\frac{(\cos\varphi'-\cos\psi)(\cos\varphi'-\cos\varphi)+(\sin\varphi'-\sin\varphi)(\sin\varphi'+\sin\psi)}{2-\frac{1}{2}(\cos\varphi-\cos\psi)^2-\frac{1}{2}(\sin\varphi+\sin\psi)^2}$$

Evidently small errors in the cosines will exert very little influence when the ratio of $\mathfrak{h}'$ to $\mathfrak{h}$ is computed from this formula. The formula (23) itself can be derived in like fashion by considering the projections of a symmetrical linkage of four equal sides. Connecting A and P, one side of this linkage, AR, is perpendicular to the incident ray, and the next (RS), which is perpendicular to the antiparallel of the refracted ray with regard to PC, terminates on the bisector of the angle PCA; the other two are images of the first two in this bisector. On taking the ratio of the projections it is found that

$$\frac{\mathfrak{h}'}{\mathfrak{h}} = \frac{\cos(\varphi'-\varphi)+\cos 2\varphi'+\cos(\psi-\varphi)+\cos(\psi+\varphi')}{1+\cos(\varphi+\varphi')+\cos(\psi-\varphi')+\cos(\psi+\varphi)}$$

The required formula can be deduced at once. The highest precision would be obtained by using it in the form

$$\frac{\mathfrak{h}'}{\mathfrak{h}} = 1+\frac{4(\sin\varphi-\sin\varphi')(\sin\varphi'+\sin\psi)}{\begin{array}{r}8-(\cos\varphi-\cos\varphi')^2-(\cos\psi-\cos\varphi')^2-(\cos\psi-\cos\varphi)^2\\ -(\sin\varphi+\sin\varphi')^2-(\sin\varphi'-\sin\psi)^2-(\sin\psi+\sin\varphi)^2\end{array}}$$


C
P
φ
φ'
$\frac{\pi}{2}+\psi$
$\frac{\pi}{2}+\phi'$
$\frac{\pi}{2}+\phi'$
S
T
A
R
Q
N
ψ

The **Author**, in reply to Mr. French, stated that an appreciable gain in speed had been found in spite of the greater number of operations involved. These could be divided into two classes, the first concerning quantities which had to be known to the same degree of accuracy as was ultimately to be attained, the second concerning quantities where a much rougher approximation could be adopted with impunity. About two-thirds of the operations related to the latter group. The calculation of the denominator in equation (23) could frequently be shortened when a paraxial ray had already been traced by noting that in the absence of aberration

$$\sin \psi_{\lambda-1} - \sin \varphi_{\lambda} + \sin \varphi_{\lambda}' = \sin \psi_{\lambda}.$$

Unless $\sin \psi_{\lambda}$ were large the square of the first quantity would not be appreciably affected by neglecting the aberration. The work could be so arranged that the individual contributions to the denominator need not be recorded separately, the denominator as a whole being calculated as the sum of two products obtained automatically by not clearing the product slide between operations. In the arrangement followed by the author, 21 paper entries would be made in calculating a ray by the new system through a cemented doublet, against 26 on the Steinheil system. The number of entries in the former system can be further reduced, but, in general, this is inadvisable, as all the numbers retained involve results of value to the computer. For example, the value of the denominator gives the relative magnitudes of the aberrations of the first and of higher orders. No detailed procedure had been suggested because this would necessarily vary with the type of machine used, as well as with the amount of information ultimately required from the calculations. Although emphasis had been laid on speed, the new system offered advantages in that the operations reflected the physical events which took place at the surface, and depended on its curvature simply, while in the Steinheil and most other systems the calculation hinged around the positions of the centres of curvature; and, further, it was usually possible to judge very well without calculation how a host of intermediate rays would be affected owing to the separation of the aberration terms from those relating to the normal refraction, a point in which the ordinary methods of calculation were notably deficient. In at least one more respect the new system enjoyed an advantage over Steinheil's. The same formulæ cover every case, and are uniformly reliable. On the other hand, with Steinheil's, a number of special cases arise in which entirely distinct formulæ must be adopted if the numerical accuracy is to be maintained. With the new method there would be no great difficulty in arranging a machine to carry the calculations automatically through a number of surfaces when the optical data and the initial ray co-ordinates were given.

Mr. Whipple's geometrical proofs were very ingenious and interesting, and the rearrangement of the denominator, though ordinarily troublesome, might be advantageous when the full accuracy of the cosine table was called for. Commander Baker's equation was of a somewhat different type from those considered in the paper. Apart from slide rule methods it was evident that linkages could be contrived to give correcting factors to the first order aberration automatically, and these might well be found to give the most rapid means of tracing rays.

XXV. *On the Mechanical Equilibrium of a Sphere of Gravitating Fluid.* *By* CHARLES H. LEES, *D.Sc., F.R.S.* (*Part of the Presidential Address to the Physical Society, February* 13, 1920.)

1. IT is now nearly 60 years since, in a letter to Joule which was read at the meeting of the Manchester Literary and Philosophical Society on January 21, 1862, Lord Kelvin showed that convection currents in the earth's atmosphere would cause the average temperature of horizontal layers to decrease with height according to a law he named that of "convective equilibrium."*

If the rate of decrease of the temperature θ with height h above the surface of the earth is given by the equation

$$-\frac{d\theta}{dh}=\frac{\gamma-1}{\gamma}\cdot\frac{g}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}1)$$

where γ is the ratio of the specific heats of air at constant pressure and volume respectively, g is the gravitational acceleration, and R the gas constant of the gas equation $pv=R\theta$, where v is the volume of 1 gram of air at the point, the atmosphere is in "convective equilibrium." If the rate of decrease of temperature with height exceeds this value, the atmosphere is unstable; if it is less, the equilibrium is stable; if it has this value, the equilibrium is neutral.

With the values of R and γ for dry air, and of g for points not far from the surface, we find that the equilibrium is stable if the decrease of temperature with altitude is less than 1°C. per 100 metres. The difference between this result and 0·6°C. per 100 metres, found from observation, was shown by Kelvin to be due to the air not being dry, and, by taking into account the heat developed by the condensation of moisture as saturated air is cooled, he showed that the theoretical value of the rate of decrease was identical with that given by observation.†

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* Kelvin, Proc. Manc. Lit. and Phil. Soc., 3rd series, II., p. 125 (1862). Abbe, in his "Mechanics of the Earth's Atmosphere" (Washington (1891), p. 172), states that the law of decrease of temperature with height was known to Espy and Henry in the United States, but he gives no references.

† Kelvin, *loc. cit.*

Kelvin's work does not appear to have been well known, for Peslin,[*] Reye[†] and Hann[‡] all dealt subsequently with the problem for dry air without referring to it.

For meteorological purposes, the effects of the presence of water vapour in the atmosphere on the vertical distribution of temperature have been traced more in detail by Hertz,[§] Bezold,[‖] Neuhoff [¶] and others.

2. In cosmological directions, Kelvin's results for a dry gas in a gravitational field of constant strength have been extended to spherical gaseous masses under their own gravitation by Homer Lane,[**] by Kelvin[††] and by Jeans.[‡‡]

In these extensions the gaseous medium is assumed to be subject to the perfect gas equation $pv=\frac{R}{m}\cdot\theta$, where p is the pressure, v the specific volume, θ the absolute temperature, R is the gas constant and m is the molecular weight of the gas, or, if the gas is a mixture, m may be supposed defined by the equation.[§§]

If for the gas when adiabatically expanded $p=A\rho^\gamma$, where γ is the ratio of the specific heats at constant pressure and constant volume, ρ is the density and A is called by Kelvin the "adiabatic constant," for neutral or convective equilibrium A is constant throughout the gas at any instant, but diminishes as time goes on,[‖‖] while for stable equilibrium A must increase from the centre of the gas to the circumference.[¶¶]

3. The inconvenience of the restriction of these propositions to media which satisfy the perfect gas equation has been felt by many workers in this field, and it is the object of the present note to show that a criterion for the stable equilibrium of any gravitating fluid or mixture of fluids may be established. By fluid is understood a medium in which the pressure is of the

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[*] Peslin, Bull. Assoc. Scientif. de France, III., p. 299 (1868).

[†] Reye, Die Wirbelsturme, Hannover (1872).

[‡] Hann, Zeit. Oester. Ges. f. Met., IX., pp. 321, 337 (1874).

[§] Hertz, Met. Zeitsch., I., p. 421 (1884).

[‖] Bezold, Sitz. Ber. K. Akad., Berlin, XLVI., p. 485, 1,189.

[¶] Neuhoff, Abh. K.P. Met. Inst., Berlin, I. (1900).

[**] Homer Lane, "American Journal of Science," LIII., p. 57 (1870).

[††] Kelvin, "Phil. Mag.," XXII., p. 287 (1887), XV., p. 687, and XVI., p. 1 (1908).

[‡‡] Jeans' "Cosmogony and Stellar Dynamics," Cambridge (1919), p. 191.

[§§] Jeans, *loc. cit.*, p. 191.

[‖‖] Kelvin, "Phil. Mag.," XVI., p. 5 (1908).

[¶¶] This fact, though implied in the reasoning of Lane and Kelvin, does not appear to have been mentioned by them. It is stated clearly by Jeans, *loc. cit.*, pp. 193-4.

hydrostatic type—that is, has the same value in all directions about a point of the medium. Solids under great pressure probably come within this category along with liquids and gases.

4. If in a fluid we draw a surface everywhere perpendicular to the direction of the gravitational force at points in it, that surface is by Clairault a surface both of equal pressure and of equal density, whether the medium be homogeneous or not. If the gravitational force has the same value at all points of such a surface, the surface itself must be spherical, and the surfaces of equal gravitational force, equal pressure and equal density are all concentric spheres.

Let r be the radius of one of these spheres, p the pressure over it, and ρ the density of the medium at it. Then, if $r+dr$, $p+dp$ and $\rho+d\rho$ be the corresponding quantities at a consecutive concentric sphere, we have for mechanical equilibrium

$$dp=-g\rho\,.\,dr \qquad (4{\cdot}1)$$

where g is the value of the gravitational acceleration at the spherical shell. This condition does not require the composition of each shell to be identical throughout its volume, but merely its density the same throughout. Nor does it imply that ρ is a continuous function of r; abrupt changes of density with radius are not excluded.

5. If a small volume, say 1 cc. of the fluid of density ρ, be removed, and be replaced by an equal volume of a medium density, ρ', the gravitational force on the substituted matter will be $g\rho'$ downwards—*i.e.*, in the gravitational sense; while the resultant pressure of the surrounding fluid on it will be $g\rho$ upwards. The resultant force downwards will be $g(\rho'-\rho)$, which will be positive, *i.e.*, downwards, if the substituted matter is denser or its specific volume less, and upwards if it is less dense or its specific volume greater than the medium immediately surrounding it.

6. Now take a small mass, say 1 gram, of the medium at r of density ρ, or specific volume v and pressure p, and allow it to change adiabatically till its pressure becomes $p+dp$. Its specific volume becomes $v+\left(\frac{\partial v}{\partial p}\right)_\phi dp$. Where the subscript ϕ implies that the entropy ϕ is kept constant. That is, $v(1-\beta_\phi dp)$ where β_ϕ is the adiabatic compressibility of the

medium at r. In like manner, take an equal mass of the medium at $r+dr$, of density $\rho+d\rho$ or specific volume $v+dv$, and at pressure $p+dp$, and change it adiabatically till its pressure becomes p. Its specific volume becomes

$$v\left(1+\frac{dv}{v}+\beta_\phi dp\right).$$

Now interchange the two equal masses. The forces acting on the two will be opposed to the interchange if the mass brought down has a larger specific volume than v, and the mass carried up a smaller specific volume than $v+dv$. That is, the equilibrium will be stable if

$$v(1-\beta_\phi dp)<v+dv$$

and $$v\left(1+\frac{dv}{v}+\beta_\phi dp\right)>v.$$

Both conditions are satisfied if

$$dv+v\beta_\phi dp>0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6{\cdot}1)$$

that is, if $$\frac{dv}{dr}+v\beta_\phi\frac{dp}{dr}>0,$$

which reduces, since $\frac{dp}{dr}=-\frac{g}{v}$

to $$\frac{dv}{dr}>g\beta_\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6{\cdot}2)$$

or the rate of increase of the specific volume of the fluid with distance from the centre at any point must exceed g times the adiabatic compressibility of the medium at that point for stable equilibrium; the two must be equal for neutral and the latter exceed the former for unstable equilibrium.

Since the composition of the fluid has not been specified in the above discussion, the condition for stable equilibrium holds for a fluid mixture of any composition whose specific volume and adiabatic compressibility are known or calculable.

7. As β_ϕ is not the compressibility which is directly measured, it may be calculated from the value of the observed isothermal compressibility β_θ by the help of the relation $\beta_\phi=\frac{\beta_\theta}{\gamma}$, where γ is the ratio of the specific heat at constant pressure to that at constant volume, in cases in which γ is directly determined.

In all cases it can be calculated from the relation

$$\beta_\phi = \beta_\theta - \frac{1}{c_p} \cdot \frac{\theta}{v} \cdot \left(\frac{\partial v}{\partial \theta}\right)_p^2 \quad . \quad . \quad . \quad . \quad (7{\cdot}1)$$

where c is the specific heat at constant pressure measured in watt units.

For a perfect gas, $\beta_\theta = \frac{1}{p}$, and hence $\beta_\phi = \frac{1}{\gamma p}$; while for liquids $\beta_\phi = \beta_\theta$ to a degree of approximation sufficient for many purposes.

8. If the entropy of a gram of the fluid can be expressed in terms of its specific volume and its pressure, which will be the case if the fluid is homogeneous, we then have

$$\frac{d\varphi}{dr} = \left(\frac{\partial \varphi}{\partial v}\right)_p \frac{dv}{dr} + \left(\frac{\partial \varphi}{\partial p}\right)_v \frac{dp}{dr}.$$

But

$$\frac{dp}{dr} = -\frac{g}{v}.$$

Hence

$$\frac{d\varphi}{dr} = \left(\frac{\partial \varphi}{\partial v}\right)_p \frac{dv}{dr} - \left(\frac{\partial \varphi}{\partial p}\right)_v \cdot \frac{g}{v}. \quad . \quad . \quad . \quad . \quad . \quad (8{\cdot}1)$$

Substituting in the condition for stable equilibrium

$$\frac{dv}{dr} > g\beta_\phi,$$

or

$$\frac{dv}{dr} > -\frac{g}{v}\left(\frac{\partial v}{\partial p}\right)_\phi,$$

the value of $\left(\frac{\partial v}{\partial p}\right)_\phi$ in terms of differentials of the entropy

$$\left(\frac{\partial v}{\partial p}\right)_\phi = -\frac{\left(\frac{\partial \varphi}{\partial p}\right)_v}{\left(\frac{\partial \varphi}{\partial v}\right)_p},$$

we have stable equilibrium if

$$\frac{dv}{dr} > \frac{g}{v} \cdot \left(\frac{\partial \varphi}{\partial p}\right)_v \Big/ \left(\frac{\partial \varphi}{\partial v}\right)_p,$$

that is, if

$$\left(\frac{\partial \varphi}{\partial v}\right)_p \frac{dv}{dr} > \frac{g}{v}\left(\frac{\partial \varphi}{\partial p}\right)_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (8{\cdot}2)$$

which is the condition that in (8·1)

$$\frac{d\varphi}{dr} \text{ is } +ve. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8{\cdot}3)$$

Thus, in a sphere of a homogeneous fluid in stable mechanical equilibrium under its own gravitation, the entropy of unit mass of the fluid must increase as the radius increases. The equilibrium is convective or neutral when the entropy is independent of the radius, and the fluid is unstable if the entropy decreases as the radius increases.

(*To be continued.*)

XXVI. *The Origin of the Elements.* By J. H. VINCENT, *M.A., D.Sc., A.R.C.Sc.* (*Head of the Physics Department, L.C.C. Paddington Technical Institute.*)

RECEIVED FEBRUARY 5, 1920.

THE study of radio-activity has provided details of the origin of a large number of elementary substances. The view that all the elements are genetically connected, the steps occurring as a result of radio-active change, is one that is in the minds of many workers, and has been for some time. In the following Paper evidence is adduced in favour of this view, which is discussed in the light of recent results. The theory is then applied to the problem of representing the atomic weights as functions of Moseley's numbers.

In Figs. 1 and 2 the atomic weights of the 83 elements of the international list* are plotted against Moseley's numbers. These last numbers, in a few cases, are not the exact numbers found by Moseley, but the present accepted numbers, which indicate the total positive charge on the atomic nucleus. In all cases the international atomic weight $(0=16)$ has been plotted, and in most cases the atomic weights of isotopes, either experimentally known from Aston's results,† or assumed to exist to fit in with the view taken in this Paper, are also inserted.

Isotopic Arrays and Classes.

When the atomic weights have been reinvestigated by means of positive-ray analysis, and by the chemical determination of the atomic weights of elements in pure mineral species, it will be possible to draw a diagram in which the atomic weights of the isotopes of each element are plotted as the multiple-valued function of the nuclear charge. The curve which is now obtained by plotting the international atomic weights against Moseley's number is that of a single valued function. This is merely because the atomic weights used are the automatically weighted means of the relative masses of atoms of isotopes.

Thus at present we plot 24·32 as the atomic weight of the element whose atoms have a nuclear charge of 12 electronic units of positive electricity. This is probably the weighted mean of the isotopes, and instead of a single point with 24·32 as ordinate, we must think of a series of points all with 12 as abscissa, each point representing an isotope of magnesium.

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* "Chem. News," January 23, 1920.

† "Nature," November 27, December 18, 1919, and March 4, 1920; "Phil. Mag.," April and May, 1920.

The mean value is replaced by an array of values. This isotopic array may vary in extent over an unknown range.

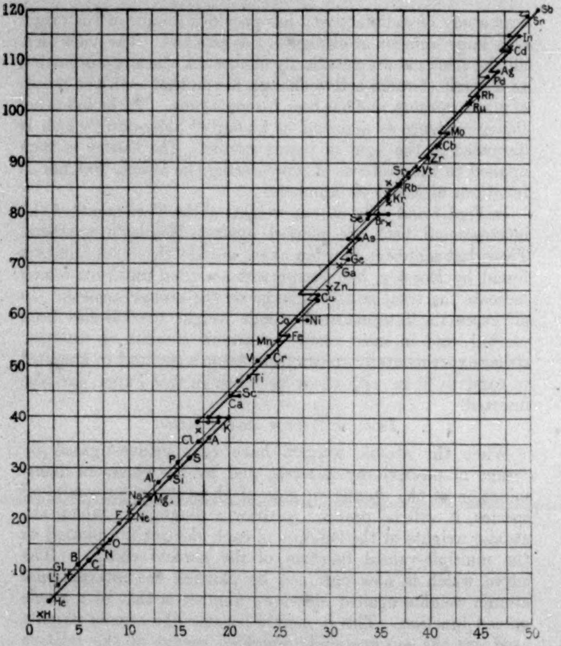

FIG. 1.

In the case of lead it is known to vary over eight units of atomic weight; with chlorine and neon over two units, while krypton has six isotopes varying in relative mass from 78 to 84. What numerical restrictions the separate terms of an isotopic array are subject to is unknown, but with the exception of hydrogen they all seem to be whole numbers, or to approach very closely to whole numbers.

Aston finds that hydrogen is a "pure" element of atomic weight 1·008 when 0=16. Hydrogen is thus in a class apart ; other isotopes are grouped into classes indicated by the remainders found by dividing the atomic weights of the isotopes by four.

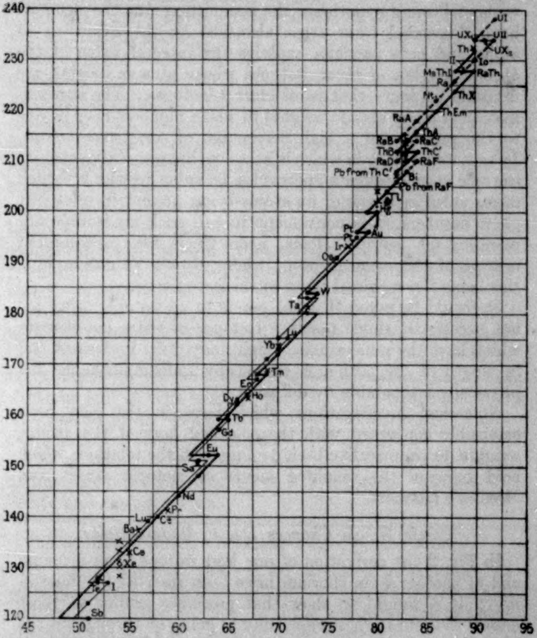

Fig. 2.

Prout's hypothesis can then be brought up to date by assuming that the atomic weights of the separate isotopes (with the exception of hydrogen) are whole numbers when oxygen is taken as 16.

Rutherford has proved that the mass of the alpha particle is very accurately four, and Aston has shown that oxygen is "pure," in the sense that it is not a mixture of isotopes. Thus, without Prout's hypothesis, the assumption that the elements have descended from other elements by the expulsion of alpha and beta particles explains the integral value of the atomic weights of those isotopes whose atomic weights are divisible by four—that is, of class 0 isotopes. The approximation of many atomic weights to whole numbers may be due to the presence of a high percentage of some one isotope. In an isotopic array in which it happens that one of the separate isotopic atomic weights approaches in value to the weighted mean value in virtue of its atoms being present in relatively great numbers, it is convenient to call such an isotope the predominant isotope. Thus, neon 20 is the predominant isotope of the element neon, whose atomic weight is 20·2, this being the weighted mean of the isotopic array.

It should be noted that the points in an isotopic array are not necessarily single points. Isotopes of the same element which have the same atomic weight may differ in descent and in length of life, although chemically indistinguishable and physically inseparable by diffusion.

Numerical investigations which have in the past been ostensibly concerned with the so-called laws of the atomic weights, have in reality been dealing with the relations which hold between the weighted means of isotopic arrays and Moseley's numbers.

Atomic Weight Changes Due to Radio-activity.

In Fig. 2 the derivation of one lead isotope from uranium and of another from thorium have been inserted for illustration; it is sought to show that processes similar to those known to occur in these cases may be going on throughout the whole range of the known elements. In the figures the unit change of atomic nuclear charge is represented on twice the scale of unit change in the atomic weight, so that the loss of an alpha particle is represented by the movement of the indicating point on the diagram downwards along a line at 45 deg. to the axes. For example, UI emits an alpha particle to become UX_1, which is four units lighter than UI, and whose nucleus carries two units less of positive electricity. Changes such as this may be termed alphatopic, and the same word may be used even when the change in the atomic weight is greater, so long as this is a multiple of four and is accompanied by a movement of two places to the left in the periodic table for every four units loss of atomic weight.

When an atom emits a beta particle its atomic weight is not appreciably affected, but it moves one place to the right in the periodic table. Such a change may be termed a betatopic one, and no confusion will arise if the word is used in a more extended sense. Thus, UX_1 gives out a beta particle to become UX_2, which, on discharging another electron, turns into UII. The motion of the indicating point on Fig. 2, representing the betatopic change from UX_1 to UII is parallel to the axis of Moseley's numbers, and two units long.

Now, the change from UI to UII involves the loss of both kinds of particles; this may be called alphabetatopy. The conditions to be satisfied by an isotope in order for it to be possibly derived by alphabetatopy from another is that its atomic weight must be less than the parent by a number of units divisible by four, and that twice the quotient is greater than the number of places the daughter element has moved to the left of the parent in the periodic table. The use of these three words to indicate the expulsion of helium nuclei, of electrons and of both kinds of particles saves circumlocution. Alternative terms would be heliotopy, electrotopy; but a combination of these two would be very inelegant.

The physical truths outlined above are well known owing to the discoveries and writings of J. J. Thomson, Rutherford, Soddy and their students.

Examples of Similar Relations Among Other Elements.

H. Collins has long since called attention to a number of pairs of elements which are related to each other chemically and by their atomic weights, as if the heavier of the two had produced the other by losing an alpha particle.*

It is well known that most of the international weights which have integral values are either exactly divisible by four or when divided by four have three for remainder. Assuming that in most cases the predominant isotope has the atomic weight of the element, they have been joined in the diagrams by lines indicating possible radio-active changes. In Figs. 1

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* "Chem. News," February 28, 1913.

and 2 the thin lines join isotopes whose atomic weights have three as a remainder when divided by four (class 3), and the thick lines join those whose atomic weights are exactly divisible by 4 (class 0).

Description of the Tables and Figures.

Paying attention chiefly to the elements whose atomic weights are given in the international list, Table I. gives a list of the elements in which an isotope of class 0 is present or is assumed to be present, those in which an isotope of class 3 occurs being given in Table II.

Omitting the intensely radio-active bodies, Table I. begins with bismuth.

Before the symbol of each element is given its Moseley's number, and after it is given the accepted atomic weight, while in the third column is given the atomic weight of an isotope of the element.

The fourth column of Table I. contains entries from Table II., and gives the isotopes of elements from that table when the same element occurs in both tables. The last column of Table II. similarly gives isotopes from I.

Alphatopic changes are indicated by a single bar at the left of the tables, betatopic changes by no bar, and alphabetatopic changes by two bars. A branch product is indicated by its inclusion in curved brackets.

Class 0 isotopes are joined by thick zig-zag lines in the figures, the thin lines passing through class 3 isotopes.

It will be seen that the atomic weights of the isotopes are usually either equal to or approach closely to those of the corresponding element. In many cases the atomic weight of the element lies between those of two isotopes of different classes. For example, the atomic weight of antimony is 120·2. The predominant isotope has an atomic weight of 120, the excess of the atomic weight of antimony above this being accounted for by the presence of a class 3 isotope of mass 123. The atomic weight of calcium is accounted for by the presence of two isotopes 40 and 44 of the same class.

Only one isotope (200) of mercury has been included in the tables; another of class 3 of atomic weight 199 alphatopically related to platinum 195 would join up with the list in Table II. Aston has provisionally given the isotopes of mercury as 197-200, 202, 204. Xenon probably has one isotope of class 0 and two of class 3, which are omitted from the tables. The

TABLE I.

Mosley's number.	—	Atomic weight of chemical element.	Atomic weight of class 0 isotope.	Atomic weight of class 3 isotope. *See* Table II. & remarks.
83	Bi	208·0	208	......
81	Tl	204·0	204	*See* also 3 lines lower
79	Au	197·2	200	......
80	Hg	200·6	200	......
78	Pt	195·2	196	195
79	Au	197·2	196	......
74	W	184·0	184	......
73	Ta	181·5	180	183
70	Yb	173·5	172	175
68	Er	167·7	168	167
69	Tm	168·5	168	171
67	Ho	163·5	164	......
65	Tb	159·2	160	159
63	Eu	152·0	152	......
64	Gd	157·3	152	159
62	Sa	150·4	148	151
60	Nd	144·3	144	......
58	Ce	140·25	140	......
52	Te	127·5	128	127
51	Sb	120·2	120	123
48	Cd	112·40	112	115
47	Ag	107·88	108	......
42	Mo	96·0	96	......
38	Sr	87·63	88	87
36	Kr	82·92	84	83
34	Se	79·2	80	79
35	Br	79·92	80	......
(36	Kr	82·92	80	83)
29	Cu	63·57	64	63
26	Fe	55·84	56	......
24	Cr	52·0	52	......
22	Ti	48·1	48	......
20	Ca	40·07	44	*See* also 4 lines lower.
(21	Sc	44·1	44	47)
18	A	39·9	40	......
19	K	39·1	40	39
20	Ca	40·07	40	......
18	A	39·9	36	......
16	S	32·06	32	......
14	Si	28·3	28	......
12	Mg	24·32	24	......
10	Ne	20·2	20	......
8	O	16·00	16	......
6	C	12·005	12	......
2	He	4·00	4	......

TABLE II.

Moseley's number.	—	Atomic weight of chemical element.	Atomic weight of class 3 isotope.	Atomic weight of class 0 isotope. *See* Table I. & remarks.
78	Pt	195·2	195	196
76	Os	190·9	191	......
73	Ta	181·5	183	180
70	Yb	173·5	175	172
71	Lu	175·0	175	......
69	Tm	168·5	171	168
68	Er	167·7	167	168
66	Dy	162·5	163	......
64	Gd	157·3	159	152
65	Tb	159·2	159	160
62	Sa	150·4	151	148
57	La	139·0	139	......
52	Te	127·5	127	128
53	I	126·92	127	......
51	Sb	120·2	123	120
50	Sn	118·7	119	......
48	Cd	112·4	115	112
49	In	114·8	115	......
46	Pd	106·7	107	......
45	Rh	102·9	103	......
40	Zr	90·6	91	......
38	Sr	87·63	87	88
36	Kr	82·92	83	80
34	Se	79·2	79	80
32	Ge	72·5	75	*See* also 2 lines lower
33	As	74·96	75	......
32	Ge	72·5	71	......
29	Cu	63·57	63	64
27	Co	58·97	59	......
(28	Ni	58·68	59)	......
25	Mn	54·93	55	......
23	V	51·0	51	......
(21	Sc	44·1	47	44)
17	Cl	35·46	39	*See* also 2 lines lower
19	K	39·1	39	40
17	Cl	35·46	35	......
15	P	31·04	31	......
13	Al	27·1	27	......
11	Na	23·00	23	......
9	F	19·00	19	......
5	B	10·9	11	......
3	Li	6·94	7	......

two isotopes of argon found by Aston have places in Fig. 1 and Table I.

In drawing up the tables alternative methods of connection have not been given. Wherever alphabetatopic changes occur (shown by a double bar at the side of the tables), these alternative paths may occur in the figures, and thus isotopes of the same mass but different pedigree may exist.

All the isotopes of all the elements to be found in Figs. 1 and 2 are not given in the tables, only those which are helpful in explaining the atomic weights of the elements. Thus, in the case of tungsten, only one isotope is mentioned in the tables, this is 184, being of the same value as the atomic weight of the element; the other possible isotopes which can be found by the inspection of Fig. 2 are omitted.

It is a very remarkable circumstance that in the international list of well-determined atomic weights there are so few cases which suggest betatopy. This may be because the substances emitting beta rays are short lived when compared with those emitting alpha rays. The occasional occurrence of a short-lived substance emitting slow alpha rays is suggested by the absence of members from alphatopic series. Thus, the series at the end of Table I. beginning with Ca 40 has no known member corresponding with Moseley's number 4. This is a short-lived isotope of glucinum, which splits up into two helium atoms.

In Figs. 1 and 2 the atomic weights of elements not mentioned in Tables I. and II., which have atomic weights suggesting the presence of predominant isotopes of class 1 (with remainder 1 when divided by 4) and of class 2 (with remainder 2), are shown by crosses; the same sign is used when the atomic weight differs appreciably from that of its isotopes.

Missing Elements.

There are a few Moseley numbers (75, 72, 61, 43) below those of the well-known radio-active elements, isotopes corresponding to which are not known. Their fewness is simply explained on the theory adopted in this Paper. If an element has no long-lived representatives from any of the lines of descent, then it would be so rare as to be difficult to separate and identify by the older methods. That the isotopes from the different families should all happen to be short lived is unlikely, and thus such cases are rare. Even such substances will, however, be tracked down when positive ray analysis is applied to mineralogy. This method, due to Sir J. J. Thomson, has been recognised by him from the first as an exceedingly powerful instrument of research. It is, however, one requiring elaborate equipment, and is at present only being applied in the Cavendish Laboratory. It is easy to see that very far-reaching investigations are within its power; when the full results have been gathered, figures and tables on the general plan of those accompanying this Paper will be an essential part of the doctrine of matter; but these figures and tables of the future will be drawn up without any taint of mere arithmetical ingenuity. It is quite possible, however, that, even in the light of wider knowledge, the two main lines of descent traced in the present figures will be corroborated, and more especially it is probable that the main alphatopic series of classes 0 and 3 will be experimentally verified.

Other Elements.

Tables I. and II. include the great majority of the elements not recognised as radio-active. The majority of the remaining elements fall into class 1, few having atomic weights indicating predominant isotopes of class 2. The Moseley numbers are, in general, so far apart in these two classes that it would be largely a matter of choice were figures and tables to be drawn up for them. It may be noted that a magnesium of weight 25 would be related alphatopically to chlorine 37, and, together with magnesium (24), would account for the atomic weight of ordinary mixed magnesium 24·32. This would remove the only marked deviation of the value of the atomic weights of the elements in Tables I. and II. from a value either very near to that of an isotope or lying between those of two isotopes.

Although nitrogen has an atomic weight of 14·01, Aston finds that it is a "pure" element, having only one isotope, 14. Rutherford has found that when nitrogen is bombarded with alpha rays hydrogen is probably produced. He suggests that the nitrogen nucleus consists of three helium nuclei, each of atomic mass 4, and either two hydrogen nuclei or one of mass 2.* The remnant left after three helium nuclei are subtracted from the nitrogen nucleus would be, if this loss of alpha particles were the result of radio-activity, a substance of mass 2 isotopic with hydrogen. The result of smashing up the atom by bombardment with alpha particles would probably be different from that consequent on the action of automatic disintegration.

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* "Phil. Mag.," June, 1919.

Are Most Elements Radio-active?

It may be objected that a scheme such as is outlined in this Paper is put out of court at once by the remark that, with a few possible exceptions, no radio-active elements are known whose Moseley numbers are less than 81. This objection is not, however, conclusive. It is to a large extent met by the view that a substance may be radio-active without its radio-activity being detectable by the methods at present used. Rayless changes are recognised in the subject of radio-activity, and are inferred from the behaviour of the parent and the products. If the particles are emitted with a low enough velocity, radio-active changes may proceed undetected. This is an acknowledged principle, and will be again discussed when dealing with a most valuable suggestion of Soddy.

If, however, the rate of transformation is small, even though the few particles are detectable, the investigation of such cases is not likely to attract many experimenters while problems remain unsolved in connection with the more actively disintegrating substances.

As the atomic number is lessened the energy of the emitted particle would decrease, and the products would tend to become longer lived. On the whole, then, we should expect that, as the world aged, the proportion of the lighter atoms would increase, some of the heavier atoms disappearing altogether. The ancestors of thorium and uranium seem already to have died away. These bodies were probably actively radiating, and they have not survived in sufficient quantity to be detected even by their radio-activity. The same fate awaits all the known radio-active bodies. According to the view adopted in this Paper, all other elements are likewise doomed to disappear, with perhaps the exception of the different isotopes of hydrogen and helium. The helium as formed would, according to the theory of Johnstone Stoney, tend to leave the earth owing to its gravitational force being insufficient to retain a gas of such low density. The same causes would operate in the case of the other final products of atomic katabolism, and when most of the present contents of the earth had degenerated to the various isotopes of hydrogen and helium, these would leak away until the loss of temperature by radiation and convection brought this loss to an end.

The great majority of the known radio-active substances are so rare that they could not have been discovered and studied apart from their radio-activity. As we deal with elements of less atomic number, many of these are sufficiently long lived to enable them to be investigated by ordinary chemical methods, their length of life being correlated with their feeble radio-activity. Some of the lighter elements have been regarded as radio-active by many physicists ; rubidium and potassium may be mentioned as instances. According to Fig. 1, ordinary potassium (39·10) is a mixture of isotopes of mass 39 and 40. This latter is short lived compared with the predominant isotope, and emits only beta rays. This view is not antagonistic to experimental results.

Several minerals are known which contain notable quantities of helium, but which exhibit no radio-activity. From the standpoint now being taken, these minerals are regarded as containing one or more substances undergoing atomic disintegration, the products of the changes being given off with insufficient energy to ionise gases or to produce scintillations.

Some Radio-active Processes May be Reversible.

Although the above is apparently the future history of the earth and its contents so far as can be deduced from what is known of radio-activity at present, and from the assumption that radio-activity is an almost universal property of the elements, it also assumes that radio-active processes are irreversible, and that there are no other processes which might be regarded as atomic anabolism.

The question of the possible reversibility of radio-active processes is one of great interest. Soddy has suggested* that, in view of the fact that the quantity of radium in the common rocks of the earth's crust is so great that a layer of these rocks only a few miles thick would supply the heat lost by the earth, a process of atomic upbuilding, with the necessarily enormous absorption of energy, is taking place in the interior of the earth. The assumption that the common elements are nearly all continually disintegrating increases the necessity for some theory to account for the heat thus generated. This may be met in different ways, but the most satisfactory method of meeting the difficulty is to imagine that the atomic upbuilding which Soddy suggests is sufficiently active to absorb the energy due to the disintegration of ordinary matter as well as that due to the more actively radiating substances.

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* B. A. Report, 1906, p. 130.

In a lecture before the Chemical Society, December 19, 1918,* Soddy mentioned irreversibility as one of the chief features of radio-active change, so that the processes going on in the interior of the earth do not in Soddy's opinion include reversed radio-activity, some other type of atomic upbuilding being probably implied.

From Rutherford's experiments with nitrogen, the effect of subjecting it to alpha rays seems to be that some of the nitrogen atoms are broken up and hydrogen produced. No experiment, so far as I know, has shown that the alpha particle sticks to the atomic target to produce a reversed alphatopic change. That a reversed betatopic change may be possible is shown by an investigation by C. G. Darwin.† His Paper is concerned with the orbit of a beta particle as it passes a Rutherford nucleus, and he concludes that in certain cases it becomes a spiral going right into the centre. "Numerical calculation shows that these cases should be of fairly frequent occurrence." Now, what effect will an event like this have on the atom, the nucleus of which has absorbed a beta particle? It appears not at all unlikely that the atom would keep its atomic weight sensibly unchanged, and all its chemical properties would alter.

Its new behaviour would be that of the place of one to the left of that previously occupied in the periodic table; such would be a reversed betatopic change. For instance, lithium would by such a single reversed betatopic change become an isotope of helium. It would be a monatomic inert gas having a similar visible spectrum to helium, although the wave-lengths would not be absolutely identical. Similarly sodium would become isotopic with neon, gold could be turned into an isotope of platinum, and if the process were capable of repetition, lead into gold. It may be that the results of experiments by Ramsey, Collie and Patterson,‡ in which neon was found to be produced in vacuum tubes, and also the somewhat similar results of other investigations, may be explained as being examples of betatopy.

The result of adding negative electrons to the nucleus of an atom which has a low Moseley number might conceivably be the disappearance of the substance altogether. Suppose the nucleus of a helium atom consists merely of two positive electronic charges, and owes all its inertia to the concentration of these charges, then, if it received two negative electrons, these might cause it to vanish, an equivalent amount of energy being radiated into space. With hydrogen on the same view a single addition of a negative electron might cause annihilation.

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* "Chem. News.," February 21, 28, and March 7, 1919.

† "Phil. Mag.," February, 1913.

‡ "Chem. News," February 14, 1913.

Difficulty with Respect to the Separate Classes.

Another possible objection may be that no general scheme of the origin of the elements has been attempted.

The answer to this is that such a scheme would be premature. My main object is far less ambitious, in that I attempt to support the view that, granted the possibility of the unrecognised emission of alpha and beta particles, this will account for the existence of the great majority of the dominant isotopes of the elements. I freely admit that there are very grave difficulties in going much more into detail at present.

One trouble is that the various hypothetical radio-active families corresponding to the classes of isotopes cannot be linked up together by alphatopy and betatopy alone, although these are at present the only known methods of atomic change.

This difficulty, with perhaps its solution, is referred to by Soddy in his Chemical Society lecture. I give his own words: " A heavy atom like oxygen, for example, if expelled as a radiant particle, might not attain sufficient velocity to ionise gases, or, even if it did, the range over which the ionisation would extend, as we know from the ionisation produced by the recoil atoms, would be extremely small." After commenting on the fact that hydrogen is never emitted, he goes on: " It has always seemed to me a possibility that some genetic connection may exist after all between thorium and uranium, although I have never been able to frame even a possible mode of so connecting these two elements. With a difference of atomic weight of six units, it is impossible to pass from one to the other by addition or expulsion of helium atoms alone." Now it may be that in the above quotations we have first a solution and then an example of the type of problem to which it is the key. If the whole of the elements are to be linked up together in a connected series of radio-active families, then it is essential to assume that other than alpha and beta particles are emitted. If we reject the hydrogen nucleus, because if it were ejected it would probably have been already discovered, then we must assume the existence of at least one radiant particle whose weight is an odd number. But for a similar reason to that for the rejection of hydrogen we must also reject other nuclei of low atomic weight. This forces us to assume provisionally that there is at least one other type of emitted particle of odd atomic weight greater than 4, the justification for this assumption being that such particles may be discharged from members even of actively radiating series without having been detected.

Applications of the Theory.

One of the tests of the usefulness of a theory is that it should suggest further experiments. One such experiment is a search for the reproduction of helium in the contents of minerals freed from helium.

Lord Rayleigh found* that beryl, although inactive, contained a very considerable quantity of helium, and he suggested that this was a case of rayless change. It might be possible in this or other suitably selected cases to detect the gradual reformation of helium after freeing the materials from all detectable traces of the gas.

Another consequence of the theory is that, since atomic katabolism involves the liberation of energy, the steady temperature attained by a body in an enclosure whose walls are at constant temperature would, in general, differ from that of the enclosure, while thermally isolated bodies would rise in temperature. It might be that the slowness of the emitted particles is not always accompanied by a low rate of disintegration, and thus temperature effects might disclose the progress of disintegration.

It such cases it is not impossible that slowness of change might be to a large extent compensated for by the fact that the experiments would not be limited to the use of small quantities of matter. I am unaware of any accurate tests of the assumption frequently made that when bodies of different materials reach a steady state in an enclosure they are all at the same temperature.

The Atomic Weight as a Function of Moseley's Number.

It may fairly be asked of a theory that it should clear up difficulties which have attracted previous attention.

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* R. G. Strutt, "Nature," February 21, 1907.

Now, there is a considerable body of literature which has been concerned with the remarkable approximation to regularity in the atomic weights of the elements of low atomic weight. I think that the large majority of the very numerous and interesting connections between the chemical properties and atomic weights of these elements, and also the very close way in which the atomic weights follow simple arithmetical rules, is immediately explicable on the theory advocated. All these connections may be summed up and explained by the view that nearly all the elements of low-atomic weight have predominant isotopes which belong to well-marked alphatopic series of classes 0 and 3.

Rydberg's formula is one of the best known. Comstock* gives a comparison between the formula and the atomic weights for 1907.

If M is the number in the system, the atomic weight is, according to Rydberg's formula, $2M$ and $2M+1$ alternately, beginning with helium, for which M is 2. From helium to scandium, Rydberg's M is Moseley's number, so that $2M$ gives the successive members of class 0 isotopes and $2M+1$ the successive members of class 3 isotopes.

It is possible to see a reason why a long alphatopic series uninterrupted by betatopy should start from calcium 40. This isotope consists of 10 helium nuclei held together in some unknown way. There are no beta particles present either in calcium or in its successive products, except any that may be closely united with the constituent helium nuclei. Betatopy cannot thus occur. On this view, the number of detachable beta particles is $W/2-n$ when the class of an isotope is even (0 or 2), and is $(W+1)/2-n$ when the class is odd (1 or 3).

Another matter which the present view clears up is, why the curve obtained by plotting atomic weights against the order of magnitude in a list of atomic weights should be a somewhat regular curve convex to the axis on which the order is marked.†

Moseley's numbers are nearly all occupied, and as the order of magnitude almost invariably follows the chemical order (which is that also of Moseley), the curve is not much altered if the abscissæ are Moseley's numbers. Now, the ordinates on this curve are the weighted means of the atomic weights of each isotopic array. Any indicating point on Figs. 1 and

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* "Phil. Mag.," January, 1908.

† Vincent "On a General Numerical Connection between the Atomic Weights." "Phil. Mag.," July, 1902.

2, so far as is known at present, can move in only two ways. It can move two units to the left or four down; it can move one place to the right.

These two steps can be taken in any order or number when Moseley's number is not small. The mean position of the points of the array must at first descend by paths steeper than that due to alphatopy alone.

The tendency to discharge beta particles becomes less as the number of these decreases, so that the curve gets less steep as the mean atomic weight decreases. This will go on until betatopy becomes no longer possible, as in the case of the nucleus of calcium (40), when the remainder of the descent will be accomplished by the expulsion of alpha particles. The smoothed curve of the path of descent of an isotope of one class will thus be steep when the mass is great; it will get less steep owing to the decrease in the number of beta particles, and will join the straight line given by the final alphatopic series.

The smoothed curve of the mean atomic weight regarded as a function of Moseley's number will have the same general character.

Minet* noted that the curve obtained by plotting the atomic weights as a function of the order of magnitude consisted of two parts, a small first part comprising the elements from helium to calcium, for which $y=1{\cdot}985x$, and the second part given by $y=x^{1{\cdot}23}$. The general explanation of relations such as the above is clear. The straight line is a consequence of alphatopy, the curved one of alphabetatopy; the point where they meet indicates the exhaustion of the supply of beta particles.

In Fig. 3 the atomic weights are plotted against Moseley's numbers. On the scale used the line $y=2x$ is at 45 deg. to the axis, and is shown on the diagram.

Aston's provisional values for isotopes are indicated by crosses, those he regards as certain by circles, while the accepted atomic weights are given by dots.

The straight line is that upon which class 0 isotopes fall when their supply of beta particles is exhausted. A parallel straight line drawn one unit higher is that upon which the class 3 isotopes fall when they enter upon their final alphatopic series of changes.

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* "C. R.," 144, 8, 1907.

The curved line in Fig. 3 is the graph of

$$W_n = pn^q$$

where W_n is the atomic weight and n is Moseley's number; p and q are constants having the values 1·36 and 1·14 respectively.

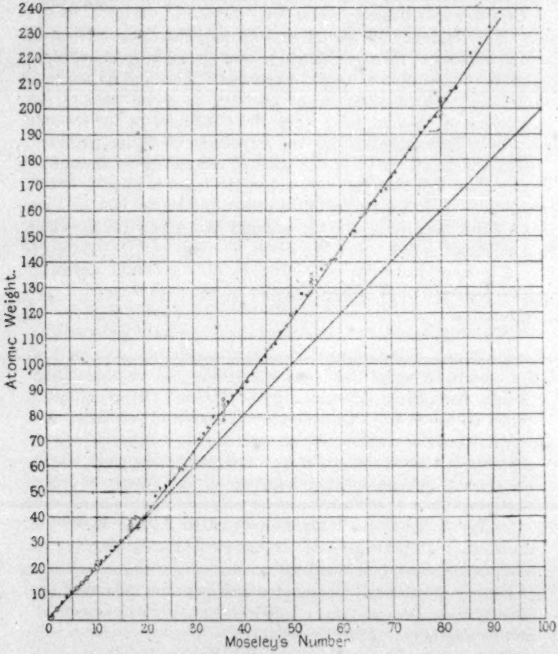

FIG. 3.

This curve cuts the straight line

$$W_n = 2n$$

near the value $n=16$. The portion of the curve below the straight line is not shown in the diagram.

Summary.

The atomic weights are regarded as the weighted mean values of the atomic weights of the isotopes of the elements; but it is assumed that, as a rule, the atomic weight is near that of some one isotope. Figures and tables are drawn up showing how this accounts for the values of a large number of atomic weights, if one also assumes that the weights and positions in the periodic table of any isotope are conditioned by laws similar to those holding in the recognised radio-active families. The elements are all supposed to be derived from parent elements by processes known to occur in actively radiating families; but their radio-activity is not in general detectable by the usual means owing to the velocity of expulsion of the particles being low.

The possibility of the reversibility of some radio-active processes is regarded favourably. The various difficulties in connection with the views advocated are discussed, and some suggestions for experiments are made. Finally, the theory is used to explain the so-called laws of the atomic weights of elements of low atomic weight, and the shape of the curve obtained when the atomic weights are plotted against Moseley's numbers.

Note added June 15, 1920.

Since this Paper was sent to the Society, Aston has published a letter in "Nature" (March 4, 1920) and two Papers ("Phil. Mag.," April and May, 1920). In revising the manuscript I have kept Aston's results in view.

DISCUSSION.

Prof. A. O. Rankine (partly communicated): During the last few months discovery in physics has been making very rapid strides, particularly in relation to the dimensions and constitution of atoms. There are, in addition to the remarkably copious information regarding isotopes obtained by Dr. Aston, two other equally fruitful researches, the results of which have been recently published. I refer to the connection between atomic dimensions and atomic number revealed last month by Prof. W. L. Bragg at the Royal Institution, and to Sir E. Rutherford's discovery—announced in the Bakerian lecture—of a new isotope of helium obtained by disintegrating oxygen. In such circumstances, Dr. Vincent may be regarded as rather bold in having presented his Paper when he did. But I have read it with very great interest, and feel convinced that, although the views expressed will have to be modified in details as discovery proceeds, these new discoveries are themselves most likely to point to connections between the elements on a broad basis similar to the lines indicated by Dr. Vincent. It may, indeed, turn out that the changes suggested by the Author are not actually spontaneous or radio-active, but that they may be effected by the supply of energy from outside. There is just one criticism of Dr. Vincent's procedure which I feel bound to make. I do not see what justification there is in assuming that every element which is a mixture of isotopes is bound to have a predominant isotope which has an atomic weight equal to the integer nearest to the mean atomic weight of the mixture. There is, indeed, definite evidence that this is not always the case. I am not sure whether Dr. Aston actually refers to it in his recently published Papers, but he has told me, in reply to a question on this very point, that in the case of bromine, the accepted atomic weight of which is 79·92, there is no isotope of weight 80, but that bromine is a mixture in very nearly equal proportions of two isotopes 79 and 81. Thus there is no bromine of zero isotopy, as assumed by Dr. Vincent. In conclusion, I would like to ask the Author whether the new isotope of helium—*i.e.*, an atom of mass 3 and charge 2—which Sir E. Rutherford has succeeded in ejecting from oxygen by alpha ray bombardment, will be found useful in explaining the end products of the radio-active changes contemplated, particularly those specified as class 3 isotopes.

Sir W. **Bragg** asked if any relationship similar to those for classes 0 and 3 held among elements of classes 1 and 2. He observed that, where lines of descent were given, the differences in atomic weight were nearly 4 or multiples of 4—*e.g.*, rhodium and palladium. This appeared to bear out the theory.

Dr. **Vincent**, replying to Prof. Rankine, said it was purely an assumption on his part to take the predominant isotope as always existing. It was very rarely indeed that it did not fit in with the facts. He had purposely kept off the subject of end products in the meantime. In reply to the President, there were some well-marked alphatopic series among classes 1 and 2.

XXVII. *An Electrical Hot-Wire Inclinometer.* *By* J. S. G. Thomas, *M.Sc.* (*Lond.*), *B.Sc.* (*Wales*), *A.R.C.S.*, *A.I.C.*

(Communicated by D. Owen, D.Sc.)

Received March 30, 1920.

In a recent communication to the Society,[*] the author described a type of directional hot-wire anemometer, consisting essentially of two equal, fine, platinum wires arranged parallel, one behind the other, transversely to the direction of flow of the fluid whose velocity was to be measured. The wires constituted two arms of a Wheatstone bridge, in which a constant current of from 1 to 1·5 ampère was maintained, and it was shown that the galvanometer deflection was reversed on reversing the direction of flow of the fluid past the fine platinum wires. Attention was directed in the Paper to certain heating and cooling effects experienced by the wires due to the passage of the stream of fluid, and to the free convection currents rising from the wires. This subject of the free convection current is treated in considerable detail in a further Paper by the author.[†] The present Paper treats of certain characteristics of a type of hot-wire inclinometer closely resembling the directional hot-wire anemometer referred to above, the free convection currents from the wires producing the cooling and heating effects to which the respective wires are subjected.

Experimental.

For the purpose of the present work, a hot-wire inclinometer was constructed as shown diagrammatically in Fig. 1. A closed chamber, *CE*, was formed of a cylindrical brass tube about 2 cm. in diameter and about 5 cm. long, closed at its ends by the screw caps *C* and *E*. Two platinum wires, *A* and *B*, about 0·1 mm. diameter, were inserted into the chamber in the manner shown, about 1 mm. apart. The wires were as nearly as possible equal in length to the diameter of the tube, and were electrically aged by passage through them of a current of about 1·5 amperes for two hours. They were attached to copper rods by means of the minimum quantity of silver solder affording smooth and secure junctions. The method of inserting the copper rods in the ebonite plugs *Q* and *R* is shown in the figure. Further constructional details may be gathered from one of the Papers referred to above.* In the present connection, it was important to affix the wires *A* and *B* to the respective pairs of copper rods so that the wires were initially coplanar, and remained in the original diametral longitudinal section of the chamber on passing a current from 1 to 1·5 ampères through them. This was readily effected by suitably turning the appropriate copper supporting rod. For this purpose the cap *C* was removed, so that the bending of the wires owing to their expansion due to heating by the current could be observed, and increased accuracy of the setting of the wires could be secured by viewing the wires with a low power microscope. Departure of the wires from the desirable coplanar condition may be entirely prevented by mounting the wires in a spring support, so that they are kept throughout in a taut condition when an electric current is passed through them. For measurements of the greatest precision, this refinement is probably necessary, although experience with the type of mounting shown in Fig. 1 showed that if precautions as above were taken to see that the wires, initially coplanar, remained coplanar on expansion, the indications of the instrument, when employed as explained later, were remarkably constant. The wires after

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[*] Proc. Phys. Soc., XXXII., pp. 196-207, 1920.

[†] "Phil. Mag.," Vol. XXXIX., No. 233, pp. 505-534, 1920.

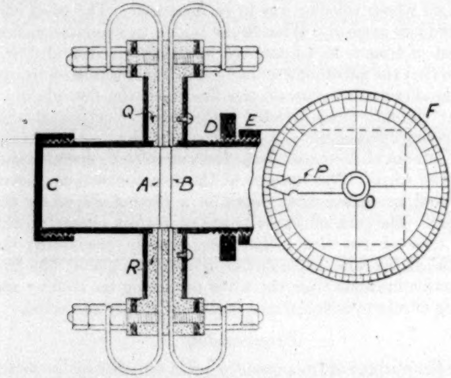

FIG. 1.

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* "Phil Mag.," 1920, *loc. cit.*

repeated heatings and coolings were found to expand and contract, so that the plane defined by them was throughout the same. The magnitude of their expansion was, of course, conditioned by the heating current employed, and could be, if desired, reduced by employing a smaller heating current. The chamber *CE* was carried by an arm capable of being revolved round a horizontal axis *O*. A pointer, *P*, moving over a divided circle of 12 in. diameter enabled the inclination of the arm to the horizontal to be determined. The divided circle was marked at 1 deg. intervals. *CE* was secured in position on this arm by means of the lock nut *D*, and could be affixed to the arm so that the plane of the wires *AB* could be inclined at any desired angle to the axis of rotation *O*. In the present experiments, the plane of the wires was in one series parallel to, and in the other at right angles, to the axis of rotation. Fig. 1 shows the case in which the plane of the wires was at right angles to the axis of rotation. The other case was obtained by rotating the chamber through 90 deg. about its own axis. The wires *A* and *B* constituted two arms of a Wheatstone bridge, the remaining arms of which were formed of a resistance of 1,000 ohms connected to one end of *B*, and a resistance capable of adjustment connected to one end of *A*. The various resistances were connected in the usual manner. The battery terminals were connected to the appropriate ends of the platinum wires, and the bridge current maintained constant by means of a rheostat inserted in the battery circuit, and could be reversed when desired. A Weston voltmeter of resistance 1,000 ohms could be inserted in parallel with *A*, so that the resistance of *A* could be calculated from the known value of the current, and the drop of potential across *A*. The resistance of *B* was calculated therefrom in the usual manner, the bridge being balanced.

Theory (a).—Axis of Rotation Horizontal in Plane of Wires and Parallel to the Wires.

Consider, first, the case shown in Fig. 2, in which the plane of the wires is parallel to the axis of rotation *O*. When *O*, *B* and *A* are in the same horizontal plane as represented in the figure by O, B_0, A_0, the wires are heated by the electric current and cooled by the free convection currents rising from them. The adjustable arm of the bridge is adjusted until balance of the bridge with the wires in the position B_0, A_0 results, the currents being adjusted by means of a rheostat to any desired value. If, now, the arm is rotated through 90 deg., so that the wires take up the position B_{90}, A_{90}, it is clear that if the current is maintained at its previous value the balance of the bridge will be upset, owing to the fact that the heating effect experienced by A, due to the hot ascending free convection current from B is greater in the position represented by $OB_{90}A_{90}$ than is the case in the horizontal position of the arms represented by OB_0A_0. This increased heating of A in the position $OB_{90}A_{90}$ necessitates a greater heat radiation from A to B and an increased free convection current from B to A.

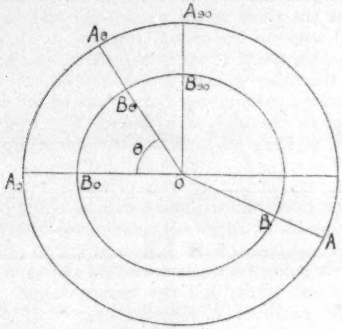

FIG. 2.

In upper semi-circle, Fig. 2, A is the hotter; in lower semi-circle, B is the hotter. There is symmetry of the values of the respective resistances about the vertical diameter through O.

The nett result upon the temperature of B is determined by the relative magnitudes of the free convection cooling effect, and the heating effect due to radiation. It is clear that, on rotation of the arm from the position OB_0A_0 to that represented by $OB_{90}A_{90}$, A comes increasingly within the heating effect of the free convection current from B. B likewise experiences an increased cooling effect due to convection, and the balance of the bridge is correspondingly upset. With passage of the arm through the position represented by $OB_{90}A_{90}$, a condition of affairs which may be best described as the mirrored image in $OB_{90}A_{90}$ of those to the left of this line is passed through. The arm rotating from 180° to 360°, a condition of affairs in which B is subjected to the heating effect of the free convection current from A is attained. It is clear, therefore, that while the condition of heating or cooling of B and A will be symmetrical about the vertical line through the 90°-270° position of the rotating arm, a symmetrical condition of affairs as concerns the resistances of the respective wires is not to be anticipated about the 0°-180° line.

(b) *Plane of Wires at Right Angles to the Horizontal Axis of Rotation of Arm.*

The initial positions of the wires are represented in Fig. 3 by B_0B_0' and A_0A_0', the wires being initially vertical. In this position it is clear that if the wires are identical in all respects, each is subjected to the same heating effect due to the free convection current due to the electric current in the other wire. When the arm carrying the wires is rotated through an angle 0, the wires take up the positions indicated by A_1A_1', B_1B_1'. Omitting for the present all reference to alteration in the heating or cooling of either wire due to its own free convection current, in the position A_1A_1' of the wire, $A_1'C$ is immediately above the portion B_1D of the wire B_1B_1', and is

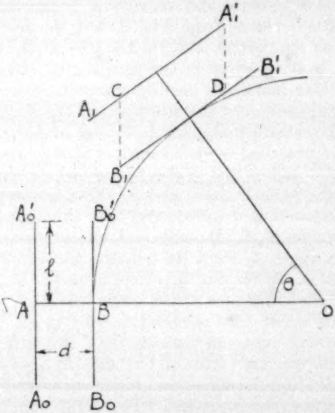

FIG. 3.

subjected to the heating effect of the free convection current arising therefrom. The portion A_1C of the upper wire is not subject to the full effect of the free convection current arising from the lower wire. The portion A_1C is, therefore, less heated than the remainder of the wire A_1A_1', but the high thermal conductivity of the wire tends to equalise the temperature throughout the wire. The portion A_1C does, however, receive some little heat from the free convection currents arising from B_1B_1', as this free convection current is not confined to the vertical plane extending over B_1B_1'. The spatial distribution of the free convection currents from the fine platinum wires as regards temperature and velocity is being investigated at present. It is evident from Fig. 3 that the free convection current from the portion DB_1' of the wire B_1B_1' will produce very little heating effect in the wire A_1A_1', and it is readily seen that the increased heating effect to which A_1A_1' is subjected on rotation of the arm through the angle θ from the horizontal is proportional to $\alpha \, . \, A_1A_1' - \beta \, . \, A_1C - \gamma DB_1'$, where α is the heat absorbed by unit length of A_1A_1', when vertically above B_1B_1', and β and γ are factors whose significance is clear from the remarks made above. To a first approximation we may assume that $\alpha = \beta = \gamma$, and then the increased heating effect to which A_1A_1' is subjected on rotation of the arm through an angle, θ, from its initially horizontal position is proportional to $2(l - d \cot \theta)$, where $2l$ is the length of each wire and d their distance apart. The increased temperature to which the upper wire is raised thereby is accompanied by (1) an increased radiation towards the lower wire, tending to raise the temperature of the latter, and (2) increased convection currents from the upper and lower wires tending to lower their respective temperatures. The final equilibrium conditions of the wires are determined by the balancing of such opposing tendencies operative on both wires, and, owing to the small mass of the wires, is attained practically immediately the alteration of the inclination of the arm is effected. The theory developed contemplates that any alteration of inclination of the arm is accompanied by a variation in the length of the upper wire immediately above the lower wire. Owing to the finite distance apart of the wires and their finite length, this condition is not secured until the arm has been rotated through an angle $\tan^{-1}\frac{d}{2l}$. This is, of course, decreased by increasing the length of the wires employed, and decreasing their distance apart. In the present experiments, where $d=0\cdot01$ cm. and $l=2\cdot05$ cm., the necessary angle of rotation $=\tan^{-1}0\cdot0025=1^\circ\,6'$. The remarks already made with regard to the symmetry of the values of their respective resistances about the vertical diameter through O apply equally in the present case as in the former case, and similarly as regards the lack of symmetry about the horizontal diameter.

Results and Discussion.

Tables I. and II. contain the results obtained when the pair of wires was rotated in a vertical plane about a horizontal axis. Table I. contains the results obtained for the case in which the horizontal axis of rotation was in the plane of the wires and parallel to the wires (*see* Fig. 2).

Diameter of cylindrical chamber	2·05 cms.
Length of cylindrical chamber	5·00 cms.
Length of wires employed	2·05 cms.
Diameter of wires employed	0·01 cm.
Distance between wires	0·10 cm.
Ratio arm in bridge	1000 ohms.
Current employed	1·000 amp.
Galvanometer shunt (to reduce sensitiveness)...	11 ohms
Rotating arm in zero position	horizontal.

A, wire at greater distance from axis of rotation.
B, wire nearer to axis of rotation.
Rotation clockwise wires initially as represented by A_0B_0.

Table II. contains the results obtained when the horizontal axis of rotation was at right angles to the plane of the wires. In the zero position of the rotating arm the wires were vertical (*see* A_0A_0', B_0B_0', Fig. 3).

The results contained in Tables I. and II. are shown graphically in Figs. 4 and 5. In Fig. 4 the respective deflections are plotted as ordinates against the angles of rotation of the arm from the horizontal position as abscissæ. In Fig. 5 the resistances of the wires are similarly plotted against the angular rotation of the arm. From Fig. 4 it is seen that, in the case where the axis of rotation is parallel to and in the plane of the wires, the sensitivity of the inclinometer is considerably increased when the angle of inclination of the arm to the horizontal attains the value corresponding to the point P. This is due to the fact that, at the inclination represented by the point P, the wire A experiences a greatly increased heating effect, due to the free convection current from the lower wire B. The deflection is very approximately constant at the inclinations corresponding to the points included between R and S.

TABLE I.—*Horizontal Axis of Rotation in Plane of Wires and Parallel to the Wires.*

Inclination of arm to horizontal. (Degrees.)	Deflection.			Balance resistance (ohms).		Drop of potential across outer wire *A* (volt).	Resistance (ohm).	
	To left.	To right.	Total.	Arm horizontal.	Arm inclined.		Wire *A*.	Wire *B*.
0	0	0	0	1004	1004	0·700	0·700	0·697
10	13	14	27	...	1009	0·700	0·700	0·694
20	27	28	55	...	1013	0·704	0·704	0·695
30	42	42	84	1004	1019	0·708	0·708	0·695
40	56	56	112	...	1023	0·711	0·711	0·695
50	73	73	146	...	1029	0·719	0·719	0·699
60	95	95	190	1004	1037	0·730	0·730	0·704
70	144	144	288	...	1053	0·749	0·749	0·711
80	221	219	440	...	1080	0·750	0·750	0·695
90	221	219	440	1004	1080	0·750	0·750	0·695
100	220	216	436	...	1078	0·750	0·750	0·696
110	146	145	291	...	1054	0·747	0·747	0·709
120	95	95	190	1004	1037	0·730	0·730	0·704
130	69	69	138	...	1030	0·720	0·720	0·699
140	56	55	111	...	1023	0·711	0·711	0·695
150	39	40	79	1004	1019	0·709	0·709	0·696
160	28	28	56	...	1014	0·705	0·705	0·695
170	12	12	24	...	1008	0·700	0·700	0·695
180	0	0	0	1004	1004	0·699	0·699	0·696
190	− 18	− 18	− 36	...	999	0·693	0·693	0·694
200	− 36	− 36	− 72	...	991	0·691	0·691	0·697
210	− 53	− 53	−106	1004	986	0·691	0·691	0·701
220	− 68	− 68	−136	...	981	0·692	0·692	0·705
230	− 88	− 88	−176	...	974	0·694	0·694	0·713
240	−119	−119	−238	1004	964	0·704	0·704	0·730
250	−161	−160	−321	...	951	0·705	0·705	0·741
260	−224	−225	−449	...	932	0·698	0·698	0·749
270	−227	−225	−452	1004	931	0·690	0·690	0·741
280	−225	−225	−450	...	931	0·697	0·697	0·749
290	−158	−159	−317	...	953	0·713	0·713	0·748
300	−116	−114	−230	1004	966	0·708	0·708	0·733
310	− 85	− 84	−169	...	976	0·696	0·696	0·713
320	− 65	− 65	−130	...	981	0·694	0·694	0·715
330	− 48	− 48	− 96	1004	988	0·693	0·693	0·701
340	− 34	− 33	− 67	...	993	0·694	0·694	0·699
350	− 17	− 17	− 34	...	998	0·694	0·694	0·696
360	0	0	0	1004	1004	0·700	0·700	0·697

TABLE II.—*Horizontal Axis of Rotation at Right Angles to Plane of the Wires.*

TABLE II.—*Horizontal Axis of Rotation at Right Angles to Plane of the Wires.*

Inclination of arm to horizontal. (Degrees.)	Deflection.			Balance resistance (ohms).		Drop of potential across outer wire *A* (volt).	Resistance (ohm)	
	To left.	To right.	Total.	Arm horizontal.	Arm inclined.		Wire *A*.	Wire *B*.
0	0	0	0	1006	1006	0·757	0·757	0·753
10	50	50	100	...	1022	0·764	0·764	0·747
20	101	98	199	...	1038	0·768	0·768	0·740
30	141	140	281	1006	1053	0·770	0·770	0·731
40	180	178	358	...	1065	0·768	0·768	0·721
50	205	203	408	...	1075	0·763	0·763	0·710
60	220	220	440	1006	1081	0·759	0·759	0·702
70	225	225	450	...	1083	0·750	0·750	0·693
80	224	223	447	...	1084	0·750	0·750	0·692
90	215	215	430	1006	1081	0·745	0·745	0·689
100	222	221	443	...	1083	0·749	0·749	0·692
110	225	224	449	...	1083	0·750	0·750	0·693
120	225	223	448	1006	1083	0·756	0·756	0·698
130	209	210	419	...	1077	0·763	0·763	0·708
140	182	181	363	...	1067	0·769	0·769	0·721
150	143	146	289	1006	1053	0·768	0·768	0·730
160	104	103	207	...	1040	0·768	0·768	0·739
170	50	51	101	...	1022	0·762	0·762	0·746
180	0	0	0	1006	1006	0·758	0·758	0·754
190	− 61	− 60	−121	...	987	0·750	0·750	0·760
200	−113	−113	−226	...	970	0·742	0·742	0·765
210	−164	−164	−328	1006	955	0·731	0·731	0·765
220	−207	−205	−412	...	941	0·720	0·720	0·765
230	−228	−226	−454	...	935	0·713	0·713	0·763
240	−227	−226	−453	1006	929	0·700	0·700	0·754
250	−230	−230	−460	...	927	0·695	0·695	0·750
260	−228	−229	−457	...	928	0·693	0·693	0·747
270	−226	−225	−451	1006	930	0·689	0·689	0·741
280	−232	−232	−464	...	931	0·692	0·692	0·743
290	−234	−233	−467	...	930	0·694	0·694	0·746
300	−229	−230	−459	1006	933	0·702	0·702	0·752
310	−222	−222	−444	...	936	0·710	0·710	0·759
320	−211	−209	−420	...	940	0·715	0·715	0·761
330	−162	−161	−323	1006	955	0·732	0·732	0·767
340	−112	−111	−223	...	971	0·740	0·740	0·762
350	− 60	− 60	−120	...	987	0·750	0·750	0·760
360	0	0	0	1006	1006	0·758	0·758	0·753

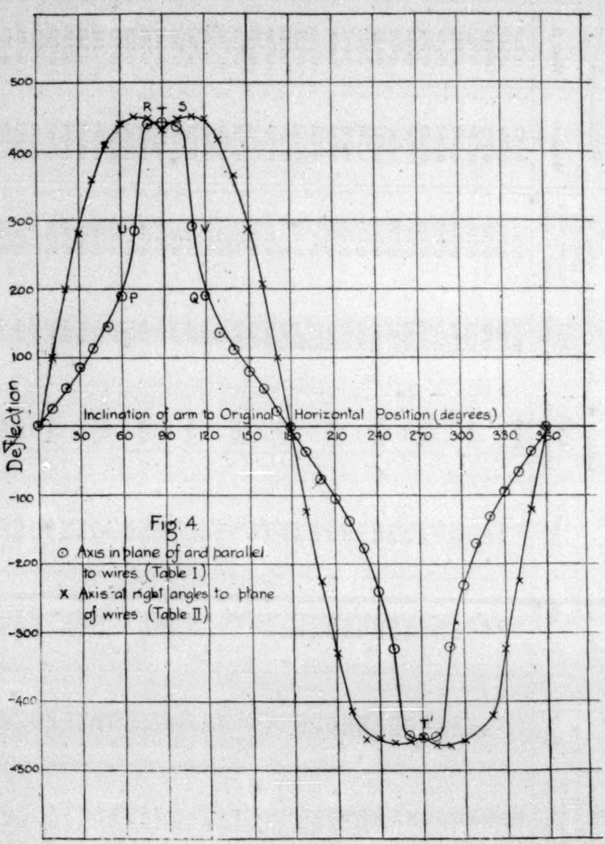

Fig. 4

Deflection
Inclination of arm to Original Horizontal Position (degrees)
500
400
300
200
100
-100
-200
-300
-400
-500
30
60
90
120
150
180
210
240
270
300
330
360
R
T
S
U
V
P
Q
T'
⊙ Axis in plane of and parallel to wires (Table I)
x Axis at right angles to plane of wires (Table II)

It will be seen from Fig. 5 that the resistance, and consequently the temperature, of the wire *A* is very approximately constant over the region represented by the points *U*, *R*, *S*, *V*. The greater deflection corresponding to *R* compared with that represented by *U* arises, as seen from Fig. 5, from the greater cooling which the lower wire *B* experiences when the upper wire *A* is in the position corresponding to *R* compared with that corresponding to *U*. This is clearly seen in Fig. 5 from the values of the resistance of the lower wire *B* represented by the portion of the curve *W*, *Y*, *Z*. A consideration of these portions of the resistance curves in Fig. 5 leads to the conclusion that the free convection current from the wire *B* is of approximately uniform temperature over a width *d* sin 10° on either side of the wire, where *d* is the distance apart of the two wires. In the present case, $d=0{\cdot}10$ cm. $2d$ sin 10° may be termed the effective width of the free convection current, and is equal to 0·034 cm., which is of the same order of magnitude as the diameter of the wire employed. Heat is therefore directly conveyed to the ascending free convection current over a width corresponding to about three times the diameter of the wire. The width is doubtless greater than the diameter of the wire owing to the fact that the

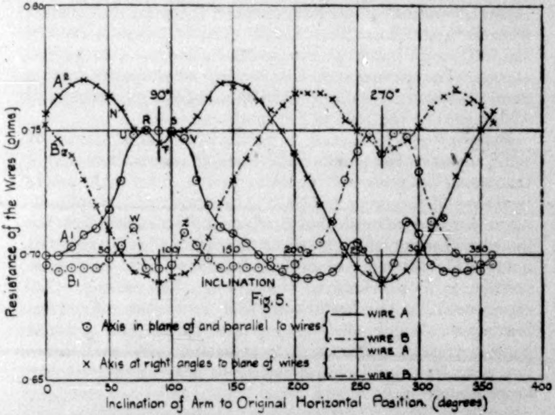

Fig. 5.

wire is circular and not "stream line" in section. The various other points connected with the deflection inclination curves in Fig. 4 can be readily interpreted by similar reference to the resistance-inclination curves contained in Fig. 5. Reference may be made to the fact that there is close symmetry between the positive and negative portions of the deflection-inclination curves in Fig. 4, about the 90° and 270° positions respectively. There is not quite such symmetry between the 0° to 180° portion and the 180° to 360° portion of the same curves. This is doubtless due to slight want of symmetry in the disposition of the wires to one another, and to the walls of the chamber containing them. In Fig. 5 there is complete symmetry between the portions of the respective portions of the resistance-inclination curves included between the values 0° to 90° and 90° to 180°. Similar symmetry is seen between the 180° to 270° and the 270° to 360° portions of the several curves. The portions of the various resistance-inclination curves between the values 0° to 180° are quite unsymmetrical with regard to the 180° to 360° portions.

Referring now to the case in which the axis of rotation is at right angles to the plane of the wires, it is seen from Fig. 4 that the device possesses its maximum sensitivity in the neighbourhood of the origin. This is in accord with the approximate theory developed above, wherein it is shown that the increased heating effect to which the upper wire is subjected on rotation through an angle θ from its initial horizontal position is proportional to $(l-d \cot \theta)$. The cooling effect experienced by the lower wire, and hence the differential heating or cooling effect experienced by the two wires, is likewise proportional to the same expression. We may therefore write H, the differential cooling or heating effect in the form $H=\mu(l-d \cot \theta)$ and

$$\frac{dH}{d\theta}=\mu d \operatorname{cosec}^2\theta,$$

and this has its maximum values when $\theta=0°$, 180°, 360°; and its minimum values when $\theta=90°$ or 270°. The form of the curve in Fig. 5 is in approximate agreement with this theory, the curve being steepest for the values 0°, 180° and 360° of the inclination, and to a first approximation, flat in the region of the inclinations 90° and 270°. One outstanding feature of the resistance curves shown in Fig. 5 deserves mention. Whereas the sensitiveness of the arrangement wherein the axis of rotation is parallel to and in the plane of the wires is due principally to the increased heating effect experienced by the upper wire of the pair, the sensitiveness of the other arrangement, in which the axis of rotation is at right angles to the plane of the wires, is due principally to the increased cooling effect experienced by the lower wire. This is made clear from a consideration of the portions of the various curves included within the region of inclinations 0° to 180° and 180° to 360°. The initial balance in the original horizontal position is effected under different conditions in the two cases. In the first case the horizontal axis of rotation, being parallel to the wires, each wire is cooled by the free convection current arising from it. In the other case, the wires being initially vertical, each is laved by its own free convection current, the resistances being therefore greater than when measured with both wires horizontal. With the wires initially vertical, it is seen from Fig. 5 that the maximum heating of the upper wire *A* occurs on rotation of the arm through 30° or 150° from the initial horizontal position. This arises from the fact that on initial rotation from the original horizontal position the heating of the wire *A* due to its own free convection current diminishes while heating due to incidence of the free convection current arising from *B* increases. Initially the temperature of *A* rises, but ultimately the cooling of *A* due to its own free convection current increases to such an extent as to equal exactly the heating effect due to incidence of the hot free convection current arising from *B*. This condition of affairs is represented by the point *N* in Fig. 5, which is such that the resistance of the wire *A* is equal to its initial resistance with the arm horizontally disposed. The resistance of the lower wire *B* is seen to decrease continuously during rotation of the arm through the first quadrant, and the approximate theory developed above is able to explain at least some of the facts, largely on account of the large magnitude of the cooling effect on the lower wire compared with the smaller thermal effects experienced by the upper wire. It is clear that if *A* and *B* were removed to a great distance from one another, the bridge initially balanced with *A* and *B* vertically disposed would remain balanced, all other things remaining the same, whatever the inclinations of *A* and *B* were, so long as they were the same. In the consideration of the deflection curves, therefore, auto-cooling of the wires due to their own free convection currents may to a first approximation be neglected, the necessary correction being merely that due to the difference in the velocity and temperature of the respective free convection currents owing to the difference in temperature of the respective wires. The variation with temperature of the velocity of the free convection current is considered in detail in the Paper by the author already referred to.

The depressions shown at the points T and T' of the curve for the case when the axis of rotation is at right angles to the wires (Fig. 4) deserve notice. It has been already remarked that the final state as regards temperature of the respective wires is determined by the production of equilibrium conditions governed by considerations of heating of the respective wires by the electric current, by radiation from each wire, and of cooling or heating by convection currents arising from each wire. In the present case, using a heating current of 1·000 ampère, the equilibrium conditions represented by T are such (as seen from T, Fig. 5) that the differential heating effect in the two wires attains a minimum value when the arm has been rotated through 90°. A reference to T, Fig. 5, shows that in the neighbourhood of 90° the resistance of the upper wire falls more rapidly than that of the lower wire. The effect can readily be interpreted in the light of the conditions of equilibrium already referred to.* It might appear at first sight that the depression represented by T originated in thermoelectric effects in the wires, the ends of which are under different thermal conditions in the case where the axis of rotation is at right angles to the wires, except when the wires are so disposed that the ends of one are vertically above those of the other. It is, however, readily seen from the symmetry of the curve in the region of T that such an explanation is entirely untenable. Apart from the elimination of thermoelectric effects from the deflection by reversal of the bridge current, it is clear that if at a small angle, θ, on one side of the 90° position, the effect of unbalanced resistances in the bridge, and any possible thermo-electric effect be represented by E and e respectively, the resultant E.M.F. in the bridge being $E-e$, at the same angle θ on the other side of the 90° position, the E.M.F. in the bridge would be represented by $E+e$ and in the 90° position by E. On rotation, therefore, from the one position to the other, the resultant E.M.F. producing deflection progressively passes through the increasing series of values—$E-e$, E, $E+e$—with consequent lack of symmetry of the calibration curve about the 90° position. The depression at

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* "Phil. Mag.," 1920, *loc. cit.*

T (Fig. 4), therefore, does not originate in thermo-electric effects. The depression is, apparently, not present in the curve Fig. 4, for which the axis of rotation is parallel to the wires. As its apparent absence from the curve might originate in its smaller magnitude in this case, and as, moreover, in this case the possibility of a thermo-electric origin is definitely ruled out, owing to the symmetrical disposition of the ends of each wire with regard to those of the other, it seemed desirable to investigate the possible existence of the effect when the wires were parallel to the axis of rotation. For this purpose the bridge was balanced with the arm rotated through 90° from the horizontal position—*i.e.*, with the one wire *A* vertically above the other *B*. The deflections when the arm was rotated through successive angles of 1° on either side of this zero position were then read, the current in the bridge being maintained constant at 1·000 ampère. The galvanometer was employed at its maximum sensitivity. The fixed arm of the bridge was adjusted to 1,000 ohms, and exact balance of the bridge with the arm in the vertical position was obtained by unplugging resistances from a box inserted in parallel with

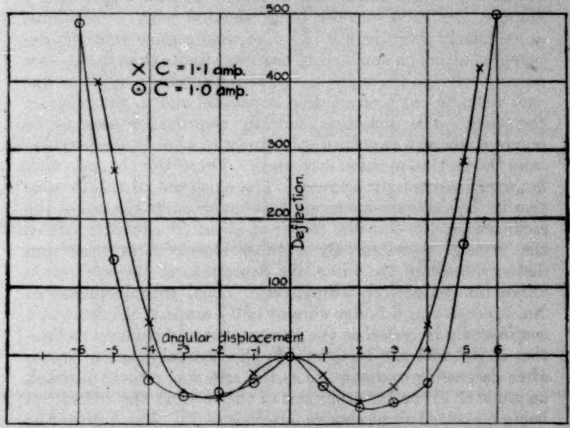

Fig. 6.

that portion of the adjustable arm of the bridge containing coils of resistance 1, 2, 3 and 4 ohms. The drop of potential across one of the wires was measured, and the bridge was balanced with the arm in each of the various positions, thus enabling the values of the resistances of the two wires to be determined. The variations of the respective resistances with rotation of the arm through a maximum of 6° were small, at the most of the order of 0·0008 ohm. The values of the deflections are plotted in Fig. 6 as ordinates against the angular displacements of the arm from the vertical position as abscissæ. Results obtained employing a bridge current equal to 1·1 ampère, the galvanometer being shunted with 1,000 ohms, are similarly given in Fig. 6. The results show that the depression, although confined to narrower limits than in the case already referred to, is present in the calibration curve for the case when the axis of rotation is parallel and in the plane of the wires. The deflection, initially negative, is seen to be reversed when an angular displacement of 4° in either direction from the vertical position is reached. Thereafter the deflection becomes increasingly positive. The existence of this depression in the calibration curve somewhat restricts the use of the inclinometer No. 2 in the region of about 5° on either side of the vertical position. By suitable choice of current, and distance between the wires, the depression in the calibration curve can be entirely eliminated. Thus, with inclinometer No. 3, employing a bridge current of 0·7 ampere, the deflection *continuously* increased as the arm was rotated from an inclination of 80° with the horizontal to the vertical position, thereafter decreasing *continuously* as the arm was rotated through an angle of 10° on the other side of the vertical, the deflections being measured at successive intervals of 1°. Fig. 7 gives the forms of the calibration curves of inclinometer No. 3, employing various bridge currents between 0·4 ampere and 1·0 ampere. The temperature to which the wire further from the axis of rotation was raised in the zero position by the respective heating currents employed is given in Table III. being deduced from the resistance and temperature coefficient of the wire in the usual manner. The temperature of the wire nearer the axis of rotation was initially very approximately the same. The axis of rotation was in every case parallel to and in the plane of the wires. The galvanometer was suitably shunted in each case, and the values of the respective shunts are indicated in Fig. 7.

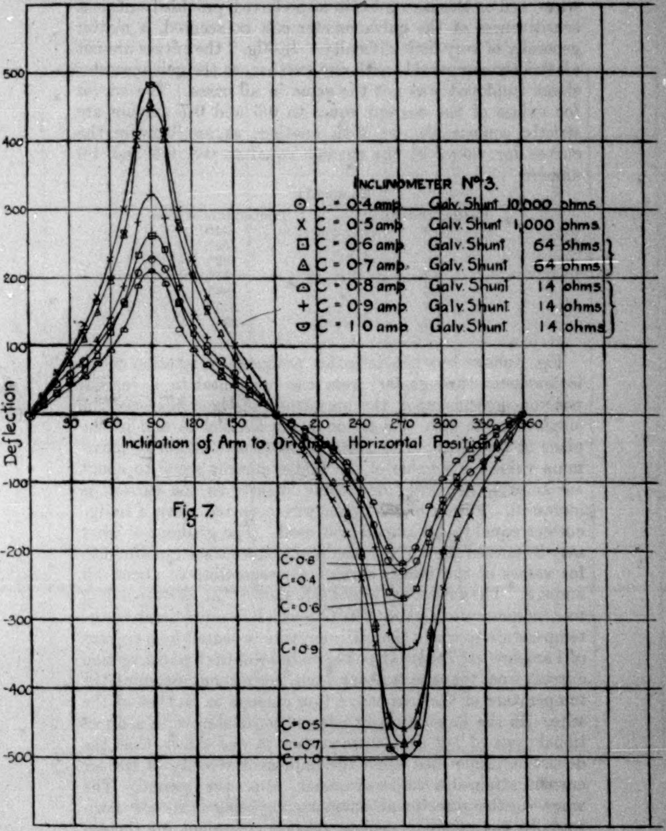

Fig. 7.

Axis in plane of and parallel to wires.

The use of small heating currents reduces the expansion of the wires, and on this account is to be preferred, provided sufficient sensitiveness of the galvanometer can be secured, a matter generally of very little difficulty. In Fig. 7 the curves are not all strictly comparable with one another, as the galvanometer shunt employed was not the same in all cases. The curves for values of the current equal to 0·6 and 0·7 ampère are strictly comparable one with another, as are likewise the curves for values of the current equal to 0·8, 0·9 and 1·0 ampère.

TABLE III.

Heating current.		Temperature of wire.
0·4	....	44°C.
0·5	....	67°.
0·6	....	92°.
0·7	....	135°.
0·8	....	186°.
0·9	....	245°
1·0	....	330°

Fig. 8 shows how the deflection produced on rotation of the inclinometer through 90° from the horizontal to a vertical position depends upon the magnitude of the bridge current employed, the axis of rotation being parallel to and in the plane of the wires. The deflection is seen to attain its maximum value for a value of the bridge current equal to about 1·5 ampere, thereafter decreasing slightly as the current is increased. Fusion of the upper wire occurred when a bridge current equal to 1·8 ampere was used. The gradient of what may be termed the current-sensitivity curve is seen to diminish for values of the bridge current corresponding to about 1·0 ampere. The author has previously shown that, corresponding to a temperature of about 300°C.—which is approximately the temperature to which the platinum wire is heated by a current of 1 ampere (*see* Table III.)—the velocity of the free convection current from the wire is about 7 cm. per second, assuming the temperature of the free convection current to be that of the wire. In the Paper already referred to, dealing with a directional type of hot-wire anemometer, it was shown that the deflection diminished when the impressed velocity of the air current attained a value of about 4cm. per second. The wires in the directional anemometer being disposed horizontally, the effective cooling velocity corresponding thereto is about 8 cm. per second. While the effective cooling current of air is, in general, differently disposed with regard

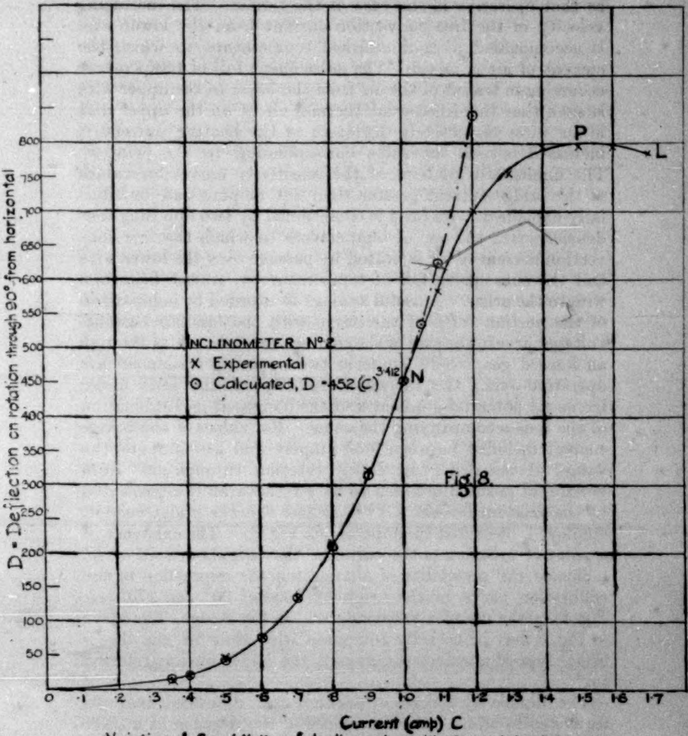

Fig. 8.

Variation of Sensitivity of Inclinometer with Current in Bridge to the plane of the wires in the case of the directional anemometer and the inclinometer, there can be no doubt that the falling off in the sensitivity of the latter for the higher values of the heating current is to be attributed to the same cause as that operative in the case of the former. The increasing velocity of the free convection current from the lower wire is accompanied by a diminished temperature to which the current of air is raised. The subsequent fall of temperature occurring in transit of the air from the lower to the upper wire is such that the differential thermal effect on the upper and lower wires respectively decreases as the heating current is increased beyond the value corresponding to the point *P*. The diminished gradient of the sensitivity curves for values of the bridge current greater than 1·0 ampère can be similarly explained. Its form is conditioned by two opposing tendencies—viz., the rise of temperature to which the free convection current of air is heated by passage over the lower wire and the subsequent fall of temperature in transit from one wire to the other. A useful analogy is afforded by comparison of the portion *NPL* of the curve with the familiar current-voltage curve in the case of the conduction of electricity through an ionised gas, where similarly two opposing tendencies are operative—viz., the increased velocity of the ions under increased potential gradient and the increased re-combination of the ions accompanying the same. For values of the bridge current included between 0·35 ampère and 1·05 ampère, the values of the deflection *D* on rotation through 90° from horizontal position are seen to be very accurately represented by the relation $D=452\ (C)^{3\cdot412}$, where *C* is the bridge current employed, measured in amperes (*see* Fig. 8). The existence of a point of inflexion in the curve in the neighbourhood of *N*, indicates the possibility of eliminating the depression in the calibration curve in the neighbourhood of 90° and 270° (*see* Fig. 4) by the use of a smaller current in the bridge. The curve in Fig. 8 may be usefully compared with that for the directional type of anemometer given in the Paper already referred to.*

The double-wire type of inclinometer described may be employed to afford optical evidence of the rotation of a body about a horizontal axis. For this purpose the Wheatstone bridge can be dispensed with, and the two wires merely joined in series with a battery and rheostat for adjustment of the current in the wires to a predetermined value, as indicated by an ammeter inserted in the circuit. The wires are viewed against a dark background. Thus, in the case of inclinometer No. 3 described above, using a current equal to 1·175 ampère, the wires being horizontally disposed one above the other, the upper wire was a very bright red, while the lower was just discernible. With rotation of the plane of the wires about a horizontal axis perpendicular to the plane of the wires, the relative brightnesses of the wires became more nearly equal, until apparent equality was reached when the wires were vertical. Thereafter, on further rotation in the same direction, the wire which was initially the lower of the pair became the higher, and consequently glowed more brightly than the other. Such a pair of wires, therefore, employed with the appropriate heating current would enable the upright or inverted disposition of an aeroplane to be readily ascertained when it is surrounded by clouds precluding the possibility of viewing the earth's surface. The device should find numerous applications in other directions, as it embodies an apparatus for measuring angles whose sensitiveness is readily adjusted within the extremely wide limits of sensitivity of the Wheatstone bridge and galvanometer employed. Thus, in Fig. 6, rotation through 1° corresponds to a deflection of about 350 mm.

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* *See* Proc. Phys. Soc., XXXII., p. 200, 1920.

The foregoing research arose out of an extensive research into the possibilities of hot-wire anemometry carried out at the physical laboratory of the South Metropolitan Gas Company, and the author desires to express his deep gratitude to Dr. Charles Carpenter for his unfailing readiness to afford all facilities for carrying out the work.

ABSTRACT.

The hot-wire inclinometer consists essentially of two fine platinum wires (diameter about 0·1 mm.) mounted parallel to one another in a closed chamber at a distance apart equal to about 1 mm. A constant current of from 0·4 to 1·5 ampere (according to the sensitivity desired) is maintained in the wires, which constitute two of the arms of a Wheatstone bridge arranged so that the bridge is balanced in the zero position of the inclinometer. If desired, the bridge may be constituted of two pairs of heated wires as above. The inclinometer wires, in the zero position, may be either vertical or horizontal or inclined at any angle to the horizontal. The indications of the instrument are dependent upon the relative heating or cooling effects experienced by the respective wires when the orientation of the wires

with regard to a horizontal plane is altered. The galvanometer deflection occurring with alteration in such orientation serves to indicate the inclination of a definitive radius vector to the horizontal, and calibration curves are given for the cases where the axis of rotation is horizontal, and (*a*) in the plane of the wires and parallel to the wires, and (*b*) at right angles to the plane of the wires. The characteristics of the calibration curves are discussed and curves are given showing how the resistances of the respective wires depend upon the orientation of the wires with regard to the horizontal plane referred to. Attention is directed to the finite width of the column of heated gas ascending from the wires, and an empirical formula is deduced expressing the maximum deflection as a function of the bridge current employed.

DISCUSSION.

Dr. **Barratt** mentioned that in a previous Paper Mr. Thomas had attributed certain effects to radiation between wires separated by about 100 times their diameter. He (the speaker) had pointed out that such an effect would be entirely negligible; and he was glad to see that the author had discarded radiation in the present Paper.

Mr. **Gosling** said that Mr. Thomas's apparatus seemed excellent for exploring the temperature in the convection stream both above and below a hot wire. He was not clear whether Mr. Thomas assumed that the narrow column of rapidly moving air extended to an equal extent below the hot wire. It would be interesting to establish if this were the case. Another interesting point was the connection, if any, between the central core of the hot column and the total width. When a hot body is standing in air there is a very definite zone of hot gas surrounding it. He had observed this recently by a modified Töpler "schlieren" method (which he described), but it was not possible to say from the appearance of the currents above a candle-flame, for instance, whether an actual discontinuity existed at the edge of the curent or not.

Mr. J. **Guild** pointed out that these and other points could readily be investigated quantitatively by placing the apparatus in one beam of a Michelson interferometer arranged to give "Contour" fringes.

Mr. F. J. W. **Whipple** referred to the practical applications such as the use of the instrument in an aeroplane. How would the results be effected by angular acceleration of the system?

Mr. F. E. **Smith** emphasised this point. It appeared that this inclinometer would give precisely the same indication as a mechanical inclinometer, such, for instance, as a suspended bob or a marble in a glass cylinder. In the two possible cases of unaccelerated flight, viz., right way up and upside down, there was never any doubt.

Dr. **Barlow** asked if the pressure of the gas affected the results given by the instrument?

Dr. **Hopwood** asked how the behaviour varied with the diameter of wire used? Some time ago he had heated two loops of fine wire in series in coal gas. At atmospheric pressure the finer of the two glowed perceptibly and the other did not. On reducing the pressure it was possible to get the thicker wire glowing more brightly than the other. Could any anomalous effects arise from such causes as this?

Mr. **Thomas**, in reply to Dr. Barratt, remarked that in the previous Paper referred to, the observed effects were not attributed wholly to the effect of radiation. Radiation was mentioned as playing some part, possibly a very minor part, in the interpretation of the phenomena referred to therein, and, contrary to the view expressed by Dr. Barratt as regards omission of reference to the effect of radiation in the present Paper, the Author felt that the radiation loss should be at least mentioned in the present connection. Conditions could readily be imagined under which radiation losses, small though they admittedly are compared with convection losses, might exercise an important rôle in the attainment of equilibrium conditions by the wires. The radiation losses from the respective wires are readily calculable from the results of Lummer and Kurlbaum's observations (Ver. Phys. Ges., Berlin, 1898, 17, 106). For polished platinum, the radiation in watts per square centimetre is given by $e=0{\cdot}514\ (\theta/1{,}000)^{5{\cdot}2}$. King (" Phil. Trans.," 1914, A 520, 422) gives a useful table of the respective radiation and convection losses from fine platinum wires of diameters between 0·0028 cm. and 0·0153 cm. for various temperatures ranging from 145°C. to 1,179°C. It is interesting to note that King, in another Paper ("Phil. Mag.," 1915, 29, 576), refers to what he terms " kinetic heating "—the heat transfer to a fine platinum wire by impact of the gas stream. This effect is normally extremely small. In reply to Mr. Gosling, the Author stated that the results obtained, more particularly those indicated in Figs. 4 and 5, pointed to the existence of what amounted to a sharp discontinuity in the ascending column of heated gas from the wire. The distribution of temperature and velocity in the convection current surrounding a heated fine platinum wire was at present being investigated. The probability is that a sharp discontinuity does not exist in the ascending column of gas below the heated wire. In a recent Paper (" Phil. Mag.," 1920, 39, 533) the Author had detailed measurements of the ratios of the thermal conductivities of various gases to that of air by comparison of the forced convection losses of heat from a heated wire in the respective gases. The wire was at a temperature of about 350°C., but the ratio of thermal conductivities found was that at about atmospheric temperature. This was interpreted as affording evidence of the existence of a skin or film of gas round the wire, which served as a medium for conducting heat from the wire to the stream of gas. Such a film possibly plays no inconsiderable part in determining the width of the hot column of gas ascending from the wire already referred to. It also affords a physical interpretation of the " effective " radius of the wire occurring in certain formulæ for the convection heat loss. The existence of such a film is supported by the results obtained by Langmuir (" Proc." Amer. Inst. Elec. Eng., 31, 1011-1022; " Phys. Rev.," May, 1912, 34; " Trans." Amer. Electrochem. Soc., 1913, 23, 293). The interferometer method suggested by Mr. Guild appeared to be admirably suited to the investigation of the temperature distribution in the ascending column of gas. The method would not, apparently, lend itself to the *direct* determination of the velocity distribution, though useful information under this head would doubtless be afforded by a knowledge of the temperature distribution. With regard to the remarks of Mr. Whipple and Mr. Smith, the Author agreed that the instrument indicated only the *apparent* direction of gravity as affected by centrifugal and other acceleration forces. He was, however, under the impression, which seemed to be rather a common one, that no means existed for discriminating between " right way up " and " wrong way up " in certain cases, and that no satisfactory indicator for the purpose existed. The device could be readily modified so as to indicate rotation about a vertical axis, and in this form might conceivably be usefully employed in experiments in wind channels for setting models accurately at any desired inclination to the wind current. The device possessed the advantage that it enabled inclinations to be indicated or recorded at any station however distant from the experimental station. The device described in the Paper enabled one to discriminate angles situated in the first and second quadrants from those in the third and fourth. A modification in which the wires were in the same horizontal plane when the radius vector was vertical, used in conjunction with the device described, would enable the actual quadrant to be uniquely determined. In reply to Dr. Hopwood, the indications afforded by inclinometers constructed of wires up to 0·2 mm. were very similar, although their sensitivities were somewhat different. Experiments in this direction are being extended at present. No anomalous effects of the nature referred to by Dr. Hopwood had been noticed in the course of the present experiments, in which the wires were throughout surrounded by air. The use of the wires in coal-gas, particularly if they are employed at even a moderately elevated temperature, is accompanied by the introduction of difficulties due to occlusion of coal-gas by the wire, and possibly by decomposition of certain hydro-carbon constituents of the gas. Such possibilities may have some bearing on the interpretation of the phenomenon observed by Dr. Hopwood. In reply to Dr. Barlow, the Author stated that the experiments had hitherto been confined to air at atmospheric pressure.

XXVIII. *The Radiation from a Perfectly Diffusing Circular Disc. (Part II.)*

By JOHN W. T. WALSH, *M.A.*, *M.Sc.*
(*From the National Physical Laboratory.*)

IN a previous Paper* an expression was obtained for the amount of flux received from a perfectly diffusing circular disc (*i.e.*, one radiating according to the cosine law) by another parallel and coaxial disc of given diameter, placed at a given distance from it. The method was also extended to determine the amount of flux received by the sides of an infinitely long coaxial circular cylinder bounded by a disc parallel to the radiating disc.

In the present Paper the results already obtained will be applied to give (i.) the amount of flux reflected to any point of the radiating disc if the non-radiating disc reflect the flux incident upon it according to the cosine law, and (ii.) the flux received at any point in the penumbra of the shadow formed when a radiating disc is partially eclipsed by a second coaxial opaque disc.

A treatment will also be given of the problem of finding the flux incident upon a disc inclined at an angle with the radiating disc. So far, the accurate solution of this problem has only been obtained when the edges of both discs form small circles of the same sphere (Part I., p. 65). A solution in the form of an infinite series will be obtained for the less specialised case when the axis of the receiving disc passes through the centre of the radiating disc, the two discs being unrestricted as to relative size.

(VII.) *The amount of flux reflected back to a radiating circular disc by a coaxial diffusely reflecting circular disc of given coefficient of reflection.* Let EF (Fig. 1) be a radiating circular disc of radius R, and let the flux emitted in the normal direction per unit area be Φ. Then if B be an element of a coaxial disc CD of radius r, the flux falling on B is

$$\frac{B}{2\pi r}\cdot\frac{d}{dr}\left[\frac{\pi^2\Phi}{2}(a^2+r^2+R^2-Q_r)\right]$$

$$=\tfrac{1}{2}\Phi B\pi[1-(a^2+r^2R^2)/Q\ ],$$

where $$Q_r=\sqrt{(a^2+r^2+R^2)^2-4r^2R^2},$$

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* "The Radiation from a Perfectly Diffusing Circular Disc (Part I.)," Proc. Phys. Soc., XXXII., 1920, p. 59.

and hence the flux received per unit area by a ring, of depth dr and radius x, is

$$\tfrac{1}{2}\Phi\pi[1-(a^2+x^2-R^2)/Q_x].$$

Therefore the flux radiated by this ring in the normal direction is, if k be the coefficient of reflection of the disc CD,

$$\tfrac{1}{2}k\Phi[1-(a^2+x^2-R^2)/Q_x].$$

Hence the flux received back from this ring by the disc EF is (by Section II. Note),

$$k\Phi\pi x dx[1-(a^2+x^2-R^2)/Q_x]\times\tfrac{1}{2}\pi[1-(a^2+x^2-R^2)/Q_x].$$

Hence the total flux received back is

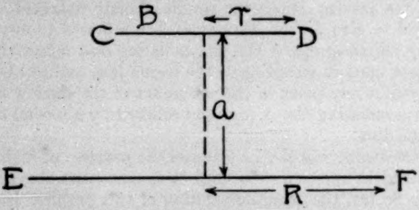

Fig. 1.—Flux Reflected Back to Radiating Disc by Coaxial Reflecting Disc.

$$\tfrac{1}{2}k\Phi\pi^2\int_0^r[1-(a^2+x^2-R^2)/Q_x]^2dx.$$

Let $a^2-R^2+x^2=2aR\tan\alpha$, then the above expression reduces to

$$\tfrac{1}{2}k\Phi\pi^2aR\int(\sec\alpha-\tan\alpha)^2d\alpha,$$

the limits of integration being

$$\tan^{-1}\frac{a^2-R^2}{2aR}$$

and

$$\tan^{-1}\frac{a^2+r^2-R^2}{2aR}$$

$$=\tfrac{1}{2}k\Phi\pi^2\Bigg[(a^2+r^2+R^2)-\sqrt{(a^2+r^2+R^2)^2-4r^2R^2}$$
$$-aR\tan^{-1}\frac{2ar^2R}{(a^2-R^2)r^2+(a^2+R^2)^2}\Bigg].$$

As a numerical example, suppose $a=R=2r=1$. The above expression reduces to $\tfrac{1}{2}k\Phi\pi^2(0\cdot1103)$. The total amount of light radiated from the disc is $\pi^2\Phi R^2$, so that the percentage reflected back is $k(5\cdot51)$ per cent.

The percentage of the radiated flux which is received by the reflecting disc is, in this case, 11·72 per cent., so that 47·01 per cent. of the flux received by the smaller disc is returned by it to the larger, if the value of k is unity.

If the discs are of equal size ($a=R=r=1$) the flux received on the reflecting disc is 38·2 per cent. of the total radiated, and the flux received back by reflection (supposing $k=1$) is 15·0 per cent. of that radiated. It is clear that this reflected flux will rapidly diminish as a increases and Fig. 2 shows graphically the value of this reflected flux for different values of a supposing $r=R$.

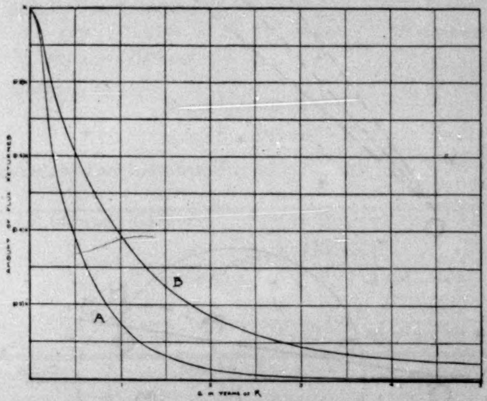

FIG. 2.—AMOUNT OF FLUX RETURNED TO EQUAL COAXIAL REFLECTING DISC AT VARIOUS SEPARATIONS, a.

Curve A.—Flux received back by radiating disc.
Curve B.—Flux received by reflecting disc.

(VIII.) *The flux received by a point in the penumbra of the shadow of an opaque disc cast by a coaxial radiating disc.* In Fig. 3, let AB be the radiating disc, of radius E, and CD a coaxial opaque disc of radius r. Let O be the point at which it is required to find the flux incident from AB.

Let the circles whose centres are L and M represent the disc AB and the projection from O of the disc CD on the plane of AB. Also let O' represent the orthogonal projection of O on this plane. Then if a and b are the distances of O from the planes of AB and CD, the effective radius of the eclipsing disc will be ar/b. Let a line be drawn through O' cutting the circles in P, Q, R and S, as shown in the figure. Then, if B be the area of the element of surface at O, and Φ the

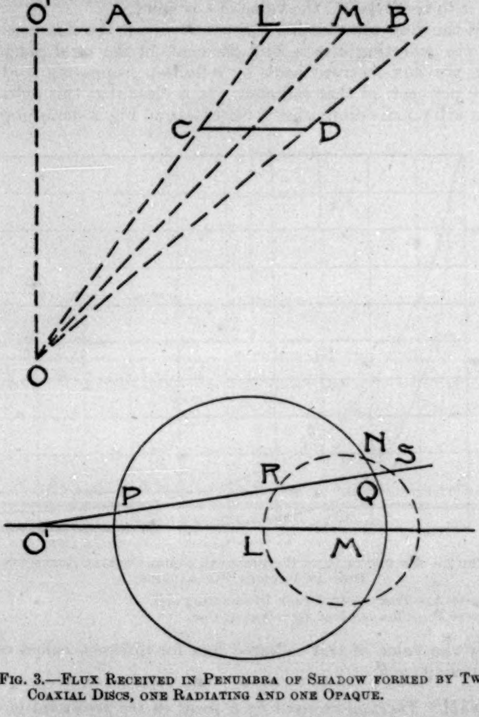

FIG. 3.—FLUX RECEIVED IN PENUMBRA OF SHADOW FORMED BY TWO COAXIAL DISCS, ONE RADIATING AND ONE OPAQUE.

normal radiation from the disc, the flux eclipsed is (*see* Part I., section I.)

$$B\Phi\int_0^\theta\left[\frac{OQ^2}{OQ^2+a^2}-\frac{OR^2}{OR^2+a^2}\right]da,$$

where $\theta=\angle^{le}LON.$

Now

$$\int_0^\theta\frac{OQ^2}{OQ^2+a^2}da=\tfrac{1}{2}\int_0^\theta\left(\frac{OQ^2}{OQ^2+a^2}+\frac{OP^2}{OP^2+a^2}\right)da+$$
$$\tfrac{1}{2}\int_0^\theta\left(\frac{OQ^2}{OQ^2+a^2}-\frac{OP^2}{OP^2+a^2}\right)da,$$

and, by Part I., section VI.,

$$\int_0^\theta\left(\frac{OQ^2}{OQ^2+a^2}+\frac{OP^2}{OP^2+a^2}\right)da=\left[a+\frac{t^2+a^2}{F}\tan^{-1}\frac{(t^2-a^2)\tan a}{F}\right]_0^\theta,$$

where

$$F^2=4a^2\rho_1^{\,2}+(t^2-a^2)^2$$
$$\rho_1=O'L \text{ and } t^2=\rho_1^{\,2}-R^2.$$

Also, by Part I., section II.,

$$\int_0^\theta\left(\frac{OQ^2}{OQ^2+a^2}-\frac{OP^2}{OP^2+a^2}\right)da$$
$$=\left[\sin^{-1}\frac{x}{R}-\frac{c}{\sqrt{c^2+R^2}}\cos^{-1}\frac{c}{R}\sqrt{\frac{R^2-x^2}{c^2+x^2}}\right]_{\sqrt{R^2-\rho_1^{\,2}\sin^2\theta}}^{R},$$
$$=\frac{\pi}{2}\left(1-\frac{c}{\sqrt{c^2+R^2}}\right)-\varphi+\frac{c}{\sqrt{c^2+R^2}}\cos^{-1}\sqrt{\frac{c^2\cos^2\varphi}{c^2+R^2\sin^2\varphi}},$$

where $R^2\sin^2\varphi=R^2-\rho_1^{\,2}\sin^2\theta.$

Similarly $\int_0^\theta\frac{OR^2}{OR^2+a^2}da$ is obtained, and hence the flux eclipsed is

$$\tfrac{1}{2}B\Phi\left[\frac{t^2+a^2}{F}\tan^{-1}\frac{(t^2-a^2)\tan\theta}{F}+\frac{c}{\sqrt{c^2+R^2}}\cos^{-1}\sqrt{\frac{c^2\cos^2\varphi}{c^2+R^2\sin^2\varphi}}\right.$$
$$-\varphi-\frac{\pi}{2}\left(\frac{c}{\sqrt{c^2+R^2}}-\frac{c'}{\sqrt{c'^2+R^2}}\right)-\frac{t'^2+a^2}{G}\tan^{-1}\frac{(t'^2-a^2)\tan\theta}{G}$$
$$\left.+\frac{c'}{\sqrt{c'^2+R^2}}\cos^{-1}\sqrt{\frac{c'^2\cos^2\varphi'}{c'^2+R'^2\sin^2\varphi'}}-{}'\varphi\right],$$

where $$t'^2 = \frac{a^2(\rho_1{}^2 - r^2)}{b^2}$$

and $$G^2 = 4a^2\frac{a^2\rho_1{}^2}{b^2} + (t'^2 - a^2)^2.$$

Also $$c = (a^2 - R^2 + \rho_1{}^2)/2a$$

and $$c' = (a^2/b^2)(b^2 - r^2 + \rho_1{}^2)/2a$$

$$R^2\sin^2\varphi = R^2 - \rho_1{}^2\sin^2\theta$$

and $$(a^2r^2/b^2)\sin^2\varphi' = (a^2r^2/b^2) - (a^2\rho_1{}^2/b^2)\sin^2\theta,$$

so that $$r^2\sin^2\varphi' = r^2 - \rho_1{}^2\sin^2\theta.$$

As an example, let $R=1$, $a=2$, $b=1$, $\rho_1=1$, $r=0{\cdot}5$,

then $t=0$, $\theta=30°$, $F=4\sqrt{2}$, $t'=3$, $G=\sqrt{89}$, $C=1$, $C'=7/4$, $\varphi=60°$, and $\varphi'=O$.

$\therefore$ Flux eclipsed $=\frac{1}{2}B\Phi(0{\cdot}179)$.

The flux received, supposing the eclipsing disc absent, is (by Section II.)

$$\tfrac{1}{2}\pi B\Phi[1 - c/\sqrt{c^2+R^2}] = \tfrac{1}{2}B\Phi(0{\cdot}92).$$

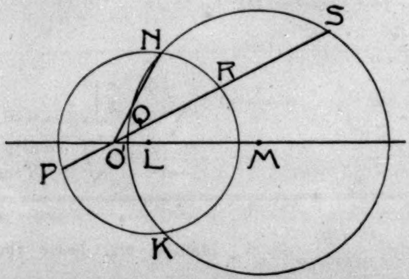

Fig. 4.—Second Case of Problem shown in Fig. 3.

Thus, approximately 19·5 per cent. of the total flux is eclipsed in this case, and the illumination at the point considered is only 80·5 per cent. of its full value, supposing the eclipsing disc removed.

It will be noticed that in the above investigation it is assumed that the second point of intersection of the line *ON* (Fig. 1) with the circle whose centre is *M* lies outside the circle whose centre is *L*. If this be not the case, as in Fig. 4, the problem may be attacked in an exactly analogous fashion, but substracting from the flux received from the uneclipsed disc *PNR*, the flux which would be received by a disc *QNS* of equal normal radiation per unit area. To the result must be added the flux received at *O* by the lenticular portion *NSKR* of this latter disc. This portion is equal to

$$\int_0^\theta \left(\frac{OS^2}{OS^2+a^2}-\frac{OR^2}{OR^2+a^2}\right) da,$$

and

$$\int_0^\theta \frac{OS^2}{OS^2+a^2}da=\frac{1}{2}\int_0^\theta\left(\frac{OS^2}{OS^2+a^2}+\frac{OQ^2}{OQ^2+a^2}\right)da+$$
$$\frac{1}{2}\int_0^\theta\left(\frac{OS^2}{OS^2+a^2}-\frac{OQ^2}{OQ^2+a^2}\right)da,$$

and $\int_0^\theta \frac{OR^2}{OR^2+a^2}da$ is obtained as before, so that the problem is solved in this case as well.

(IX.) *The flux received from a radiating disc by a second disc neither coaxial nor parallel with it, but such that its axis passes through the centre of the radiating disc.* Let *Z* be the zone of a hemisphere erected on the plane of the radiating disc *AB*, and passing through the edge of the receiving disc *CD*. Let *OL* be the axis of *CD*, passing through *O*, the centre of *AB*, and making an angle θ with *OZ*.

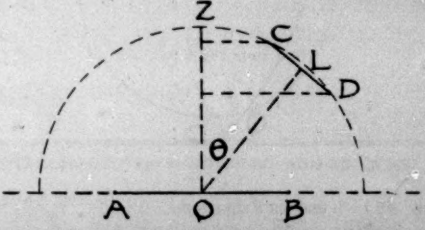

FIG. 5.—FLUX RECEIVED BY NON-COAXIAL DISC *CD* FROM RADIATING DISC *AB*.

Now suppose *CD* divided into zones by great circles, such as *ZPQ*, passing through *Z* and cutting *CD* in *P* and *Q*, and let the angle *LZP* be *a*.

The $\widehat{ZP}$ and $\widehat{ZQ}$ are given by the quadratic equation

$$\cos a = \frac{\cos r - \cos x \cos \widehat{ZL}}{\sin x \sin \widehat{ZL}}$$

where r is the angular radius of the disc CD (in terms of OZ).

Putting $\widehat{ZL}=\theta$ we have

$$\sin x = \{\cos r \sin \theta \cos a \pm \cos \theta \sqrt{\sin^2 r - \sin^2\theta \sin^2 a} / (1 - \sin^2\theta \sin^2 a).$$

$$\therefore \quad \sin \widehat{ZP} + \sin \widehat{ZQ} = 2 \cos r \operatorname{cosec} \theta \cos a / (\operatorname{cosec}^2\theta - \sin^2 a)$$

$$\sin \widehat{ZP} - \sin \widehat{ZQ} = 2 \cot \theta \sqrt{\operatorname{cosec}^2\theta \sin^2 r - \sin^2 a} / (\operatorname{cosec}^2\theta - \sin^2 a).$$

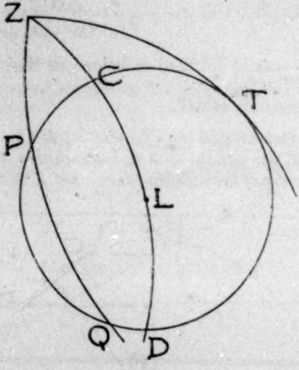

FIG. 6.—RECEIVING DISC CD CUT BY THE GREAT CIRCLE ZPQ.

Put $\sin r = l$, and $\sin \theta \sin a = m$.

Then $\sin \widehat{ZP} + \sin \widehat{ZQ} = 2 \cos r \cos a \sin \theta / (1 - m^2)$

$$\sin \widehat{ZP} - \sin \widehat{ZQ} = 2 \cos \theta (l^2 - m^2)^{\frac{1}{2}} (1 - m^2)^{-1}.$$

$$\therefore \quad \sin^2 \widehat{ZP} - \sin^2 \widehat{ZQ} = 4 \cos r \cos \theta \sin \theta \sin a (l^2 - m^2)^{\frac{1}{2}}.$$

Now that portion of the total flux radiated from AB which is received by any zone of CD cut off by two such lines as ZPQ, the angle between the lines being da is given by the expression $(1/2R^2)\{\sqrt{(R^2+1)^2 - 4R^2\sin^2 a} - \sqrt{(R^2+1)^2 - 4R^2 \sin^2 b}\}\,(da/2\pi)$ where $a = \widehat{ZP}$ and $b = \widehat{ZQ}$, and $2R = AB/OZ$. Hence if τ be the angle LZT, the flux received by the disc is

$$\{(R^2+1)/2\pi R^2\}\int_0^{\tau}\left[\{1-4R^2\sin^2 a/(R^2+1)^2\}^{\frac{1}{2}}-\{1-4R^2\sin^2 b/(R^2+1)^2\}^{\frac{1}{2}}\right]da$$

$$=(1/\pi\lambda R)\int_0^{\tau}[(1-\lambda^2\sin^2 a)^{\frac{1}{2}}-(1-\lambda^2\sin^2 b)^{\frac{1}{2}}]da$$

$$=(1/\pi\lambda R)\int_0^{\tau}\left[-\frac{\lambda^2}{2}(\sin^2 a-\sin^2 b)+\frac{\lambda^4}{8}(\sin^4 a-\sin^4 b)-\ .\ .\ .\right]da,$$

where $\lambda=2R/(R^2+1)$.

Now $(\sin^2 a-\sin^2 b)da=4\cos r\cos\theta(l^2-m^2)^{\frac{1}{2}}(1-m^2)^{-2}dm$

$$(\sin^4 a-\sin^4 b)/(\sin^2 a-\sin^2 b)=2(\cos^2\theta+\cos^2 r)(1-m^2)^{-1}-4\cos^2 r\cos^2\theta(1-m^2)^{-2}.$$

Also, the limits o to τ of a correspond with the limits o to l of m (for $\sin^2 r=\sin^2\theta\sin^2\tau$), so that $\int(\sin^{2n}a-\sin^{2n}b)da$ may always be expressed as $\int\Sigma f(r,\theta)(\sin^2 a-\sin^2 b)(1-m^2)^{-p}\,da$ where p is a positive integer and this expression

$$=\int_0^{l}\Sigma f(r,\theta)(l^2-m^2)^{\frac{1}{2}}(1-m^2)^{-p}\,dm,$$

which is always integrable for $A\int(1-m^2)^{-p}(l^2-m^2)^{\frac{1}{2}}dm$

$$=m(1-m^2)^{-p+1}(l^2-m^2)^{\frac{3}{2}}-B\int(1-m^2)^{-p+1}(l^2-m^2)^{\frac{1}{2}}dm-C\int(1-m^2)^{-p+2}(l^2-m^2)^{\frac{1}{2}}dm,$$

where $A=2(l^2-1)(p-1)$; $B=4(p-2)-l^2(2p-3)$; and $C=2(3-p)$. In this way it is found that

$$L_1=\int_0^{l}(l^2-m^2)^{\frac{1}{2}}(1-m^2)^{-2}=\frac{\pi l^2}{4}(1-l^2)^{-\frac{1}{2}}$$

$$L_2=\int_0^{l}(l^2-m^2)^{\frac{1}{2}}(1-m^2)^{-3}=\frac{\pi l^2}{16}(4-3l^2)(1-l^2)^{-\frac{3}{2}}$$

$$L_3=\int_0^{l}(l^2-m^2)^{\frac{1}{2}}(1-m^2)^{-4}=\frac{\pi l^2}{32}(8-12l^2+5l^4)(1-l^2)^{-\frac{5}{2}}$$

$$L_4=\int^{l}(l^2-m^2)^{\frac{1}{2}}(1-m^2)^{-5}=\frac{\pi l^2}{256}(64-144l^2+120l^4-35l^6)(1-l^2)^{\frac{7}{2}},$$

$$L_5=\int_0^{l}(l^2-m^2)^{\frac{1}{2}}(1-m^2)^{-6}=\frac{\pi l^2}{512}(128-384l^2+480l^4-280l^6+63l^8)(1-l^2)^{-\frac{9}{2}}$$

and thus the values of $\int_0^{\tau}(\sin^2 a-\sin^2 b)da$, &c., can be calculated.

As a numerical example, take $l=\sin r=0{\cdot}4$ and $\sin \theta=0{\cdot}5$. Then it is found that

$$\int_0^\tau (\sin^2 a-\sin^2 b)da=(0{\cdot}1386)\pi$$

$$\int_0^\tau (\sin^4 a-\sin^4 b)da=(0{\cdot}07759)\pi$$

$$\int_0^\tau (\sin^6 a-\sin^6 b)da=(0{\cdot}0440)\pi.$$

Thus, if $R=1$, the amount of flux received by the small disc is

$$\frac{0{\cdot}1386}{2}+\frac{0{\cdot}07759}{8}+\frac{0{\cdot}0440}{16}+\ldots\ldots$$

$$=\frac{1}{100}\{6{\cdot}93+0{\cdot}97+0{\cdot}27_5+\ \ldots\ldots\}$$

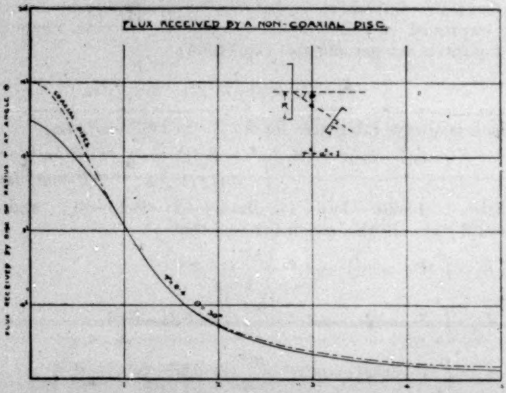

FIG. 7.—FLUX RECEIVED BY A NON-COAXIAL DISC FOR WHICH $r=0{\cdot}4$, $\theta=30°$, and $r^2+a^2=1$.

But, *in this particular case*, we know by the result obtained in Part I. of this Paper, that the total amount of flux received is $0{\cdot}083_5$, so that the sum of the residual terms in the above series is $0{\cdot}17_5$.

It is now easy to obtain the values of the above series for different values of R, for $\lambda=2R/(R^2+1)$ and the sum

$$=\frac{1}{100R}\{6{\cdot}93\lambda+0{\cdot}97\lambda^3+0{\cdot}27_5\lambda^5+\ \ldots\ldots\}$$

The values so obtained are shown graphically in Fig. 7, and superposed on this curve is the curve for the same values of r and R but with $\theta=0°$. It will be seen that the curve for coaxial discs lies above that for inclined discs everywhere, except, of course, at the value $R=1$, where they are coincident. The difference, however, is very slight except for small values of R, *i.e.*, when $R \lesssim r$.

In the case of a searchlight arc, the values of R, r and θ, may be taken for the sake of example as 0·8, 0·4 and 30°, and for this case the obscuration is only 0·1 per cent. less than for axial carbons. Since the obscuration still takes place in the solid angle embraced by the mirror there would seem to be little advantage in this method of arranging the carbons unless either (i.) the negative carbon holder is very bulky, making R smaller than the effective value of r, or (ii.) is increased to bring the obscuring object outside the angle embraced by the searchlight mirror.

XXIX. *The Construction of Thermo-couples by Electro-deposition.* *By* WM. HAMILTON WILSON *and Miss* T. D. EPPS.

RECEIVED MAY 17, 1920.

ALTHOUGH thermo-electric appliances have been used in one form or another for nearly a century, the method of constructing these has remained practically unaltered up to the present time—namely, by joining together with solder two metals suitably related to one another in the thermo-electric series.

In practice this method has several serious drawbacks, which may be briefly stated as follows :—

(*a*) There is difficulty in forming reliable joints between the dissimilar metals, which will withstand working at high temperature without deterioration.

(*b*) Continuous use of such junctions is likely to lead to deterioration of the joints, resulting in a variable behaviour or a considerable increase in resistance, while if the joint has been formed by soldering the metals together the maximum working temperature is determined by the melting point of the solder.

(*c*) Owing to the extreme difficulty in making such joints between very small wires, it is impossible to reduce the mass of the metal at the points to the extent which is necessary for some special purposes.

(*d*) The labour and difficulty of constructing a large number of junctions to operate in series is excessive, and quite impracticable when the size of the wires becomes small.

In order to carry out a certain special research the need arose for some reliable and convenient means of constructing a line of a large number of junctions all in series and having small mass.

This was first attempted by joining together small wires of metals suitably placed in the thermo-electric series, and arranging them so that the alternate junctions to be heated should lie in a straight line, and that all the junctions should be connected in series.

The constructional difficulties when the number of junctions became large and of small mass, were found to be insuperable, although a number of methods was tried. Consequently, for a considerable period the idea of carrying out this research had to be abandoned, until the facility with which certain metals can be deposited electrolytically on others naturally suggested itself as a convenient means of uniting the metals selected.

In order to put such a method into practice, it is necessary first to prepare some form of conducting core or base upon which the metals to be used for the completed couples may be deposited electrolytically. This foundation should be capable of removal after it has served for depositing the metals of the couple, or it may be composed of some material of low conductivity relative to the conductivities of the deposited metals, so that its effect on their action is negligible even if not removed. Alternatively, the cross-sectional area of this base may be made so small relatively to those which the deposited metals are required to have that its effect is negligible.

For instance, if it is desired to form a thermo-couple of copper and iron, this may be done by taking a fine constantan wire which has a high specific resistance relatively to copper or iron, and coating one-half with copper, and the remaining half with iron to any desired thickness, so that the coatings of copper and iron come in contact at the middle point of the constantan core. Then, if the thicknesses of the copper and the iron coatings are sufficient, this will act for all practical purposes in a similar manner to a couple formed by joining together a copper and an iron wire in the ordinary way.

Obviously, this method is subject to a large number of variations, some of which would only be of use in special circumstances; for example, the core may be a film or fine filament of carbon, or it may be made of a platinum or silver film on glass, mica, &c.

Consideration of the effect of relative conductivities of the core and the metals employed for coating it, will show that with suitable precautions the core itself may be used to form one of the elements of the couple.*

In order to carry this into effect due regard must be paid to the cross-sectional area of the core where it is coated, relatively to that of the coating metal, taking into account their relative conductivities, in order that the presence of the core may not be prejudicial.

The first attempt to carry out this principle was made by depositing a sheathing of copper upon a constantan wire, half way along its length, as indicated in Fig. 1, with the intention that a thermo-E.M.F. should be set up at the point where the copper sheathing ceased, when this point was raised in temperature above that of the ends. It was expected that the current would flow from the constantan wire to the copper sheath at the part heated, and that it would be practically uninfluenced by the presence of the constantan core. This was found to work successfully.

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* Since writing the above a method giving similar results has come to our notice. See British Patent No. 19758 of 1904.

Then other experiments were made by using metals situated further apart in the thermo-electric series, such as an iron sheath on a constantan core, and also an antimony sheath on a constantan core. The results obtained with these metals were disappointing, owing to the smaller difference in the conductivity of the metals necessitating a much greater volume of metal in the sheath.

It is obvious that if the mass of the sheath has to be increased unduly in order to make the presence of the core negligible, the heat capacity of the couple is liable to become too great.

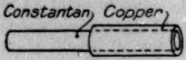


Fig. 1.-Plated Sheath.

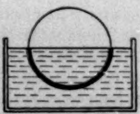
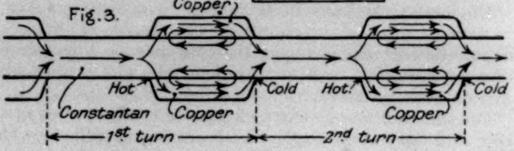


Fig. 2 Plating Bath.

Fig. 3.



After a good deal of experiment, constantan wires with either copper or silver sheaths were found to be suitable for most purposes.

Single junctions constructed by this method being found quite satisfactory, the next stage was to construct a pile consisting of lines of junctions in series. Two hundred spaced turns of 0·002 in. diameter constantan wire were wound upon a $\frac{1}{2}$ in. diameter ebonite rod. This was immersed in a copper plating bath, so that half the circumferential surface of the rod was above the level of the liquid. The wire was now plated, as in Fig. 2, thus forming a sheathing of copper on each turn for approximately half its length, the copper sheathings on all the turns ending more or less abruptly along lines diametrically opposite.

When the junctions thus formed between the constantan wire and the copper sheathings situated along one side of the cylinder were heated above the temperature of those on the opposite side, the contrivance behaved as a thermopile having a large number of junctions in series.

The path of the current in this composite conductor, at the positions corresponding to the hot and cold junctions, would be from the constantan wire to the copper sheath at the heated junction, and from the copper sheath to the constantan wire forming the next turn, at the subsequent cold junction, as shown in Fig. 3, in which the wire is represented as straightened out for clearness.

In addition to this main current which flows to the external circuit, there will be in each couple a circulating current, the path of which is diagrammatically indicated in Fig. 3. This circulating current will flow even when the external circuit is open, and its magnitude will vary with the drop of P.D. between the extreme ends of the sheathing of each couple, so that the greater the current in the external circuit, the smaller will be the circulating current in each couple.

In order to determine under a given set of conditions the effect of this circulating current, the following experiment was made: A strip of flattened 36 S.W.G. constantan wire had a piece of strip made from a 40 S.W.G. copper wire similarly flattened, soldered to its mid-point. The copper strip and one-half of the constantan strip were stretched tightly round a drum, and a heater was placed in a groove in the drum immediately beneath the soldered junction between the copper and constantan strips. The other half of the constantan strip was stretched round the drum over the copper strip, but with a thin piece of tissue paper separating the two, as seen in Fig. 4, this arrangement representing an ordinary end-to-end junction. A current was passed through the heater, and the resulting E.M.F. between the ends of the copper and constantan strips (1) and (2) was noted.

The paper insulation between the copper and constantan strips (1) and (3) was then removed, so that these two strips were in contact throughout their length, thus roughly representing a junction formed by copper plating one side of a constantan strip. With the same current through the heater, the E.M.F. between (1) and (2) was again noted, and was found to be 6·2 per cent. lower than in the first case. This result can only be regarded as approximate, since the contact between the strips could not have been so perfect as if they had been formed by plating the copper upon the constantan.

The first practical application of the principles involved was tried for the measurement of small currents. A single junction constructed as described in Fig. 1, with a 0·002 in. diameter constantan wire having a copper sheath, was stretched between two supports, and arranged to pass through the centre of a small heating coil in the form of a solenoid, the junction lying inside the heating coil without being in electrical contact with it. This was found to be very satisfactory when employed for the measurement of small alternating currents after being calibrated on direct current.

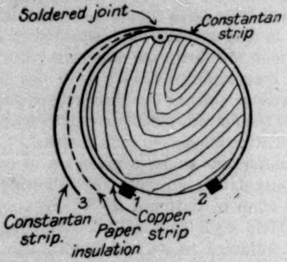


FIG. 4.

For various reasons, it appeared that for small junctions silver would be generally more satisfactory than copper for sheathing, and after frequent experiments this view was confirmed.

As it is desirable to avoid excess of metal in the silver sheath, which would result in conducting away heat more rapidly from the heated junction, beside requiring more heat to raise the larger mass of metal to a given temperature, rough trials were made, and it appeared that a suitable area for the silver sheathing was about 25 per cent. of that of the constantan core. Later, further tests were made, the results of which are shown in the curves in Fig. 5, from which it appears that the value originally arrived at was too low, and that 30 to 35 per cent. is a more suitable value.

In the test nine sets of junctions having different percentages of silver sheathing were connected respectively to a moving coil ammeter, and were arranged equidistant from a straight wire heater through which a heating current was passed.

The method was now tried for constructing junctions of large cross-section which would be capable of dealing with heavier currents. For this purpose a piece of 14 S.W.G. constantan wire was silver plated for a portion of its length, so that the cross-section of the silver sheath was about 25 per cent. of that of the constantan wire. The resistance of this junction was 0·017 ohm. On connecting the ends of the junction to the heavy brass terminals of a moving coil ammeter having a resistance of 0·01 ohm a current of 1 ampere was obtained when the junction was raised to a dull red heat by means of a spirit flame. On replacing the ammeter by a voltmeter of 500 ohms resistance, the instrument read 0·025 volt, with about the same temperature.

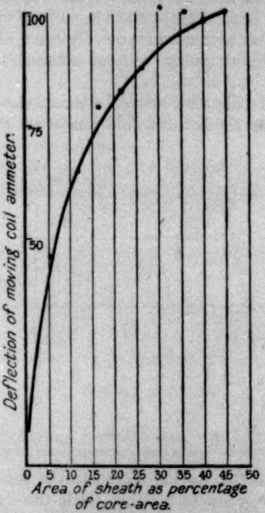


FIG. 5.

A junction made from a constantan strip, about 9/64 in. by 0·017 in., and silver plated for half its length, gave a rather better result, owing to the greater cooling surface obtained. After working this latter junction at red heat for some time, the silver sheathing was stripped back a short distance to see if the surface between the silver sheath and the constantan strip, where it was strongly heated, had undergone deterioration by oxidation. A slight trace of oxidation was observed for a distance of about 1/16 in. beyond the beginning of the sheathing, but beyond this point the surface was quite bright. This junction has been used repeatedly since, although slightly damaged by this stripping, without noticeable decrease in its effectiveness.

In order to decide what would be the most efficient distance to have between the hot and cold junctions, with a given size of round conductor, having regard to the fact that the resistance must be as low as possible, while the difference of temperature between the junctions must be as great as possible, eleven sets, each of five junctions of 0·0021 in. diameter constantan, were made up, and mounted opposite a heating wire as shown in Fig. 6.

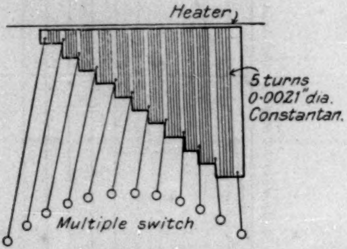


Fig 6.

When a current of 0·08 ampere was maintained through the heater, E.M.F.s were generated by the respective groups of junctions as shown in Table A. Under these conditions it appears that with that size of wire 0·45 in. is the most suitable distance between the hot and cold junctions with air cooling.

TABLE A.

Distance between hot and cold junctions.	Voltage generated.
0·5 in.	0·001628 volt.
0·45 in.	0·0016 „
0·4 in.	0·001432 „
0·35 in.	0·001132 „
0·3 in.	0·000978 „
0·25 in.	0·000872 „
0·2 in.	0·000777 „
0·15 in.	0·000519 „
0·1 in.	0·000267 „
0·05 in.	0·000287 „

As the thermal conductivity of the uncoated portions of the junctions is very much lower than that of the coated portions, while the resistance is also higher, it is necessary, in order to obtain the highest efficiency, that the uncoated portion should be made as short as possible.

To determine the best relative lengths for coated and uncoated portions the arrangement shown in Fig. 7 was made.

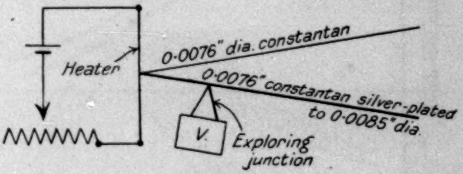


Fig. 7.

The small exploring junction was made from a 0·0016 in. constantan wire with a silver sheathing.

The results obtained are shown in Table B. :—

TABLE B.

Heating current.	Exploring junction current.	Distance of exploring junction from heater.	Ratio.
2 amp. Silver.......	2 micro-amp.	26 mm.	1·575
2 amp. Constantan	2 „	16·5 mm.	...
2 amp. Silver.......	6·2 „	17·5 mm.	1·59
2 amp. Constantan	6·2 „	11 mm.	...

This experiment shows that with a junction formed by silver-plating a 0·0076 in. diameter constantan wire to 0·0085 in.

diameter arranged horizontally in air, the temperature of any point on the uncoated wire will be the same as that of a point on the coated wire situated 1·58 times the distance from the heater.

In a similar manner the most efficient relative lengths to use under any other conditions may be determined.

In order to determine the most efficient distance between the heater and the junctions when these are not in actual contact, an experiment was made with the heater mounted horizontally opposite a line of junctions constructed as already described, the heater being arranged with a micrometer to regulate the distance between it and the junctions. The results of the two tests are shown in Fig. 8. From this it appears that under those conditions the loss of efficiency

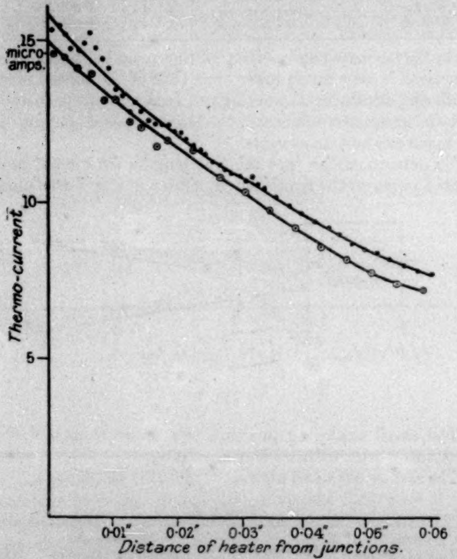


Fig. 8.

appears to be serious when the distance between the junctions and the heater is greater than 0·01 in.

For a line of junctions, the arrangement shown in Fig. 9 was found satisfactory. In this the core was of ebonite, hollowed out to facilitate cooling, and the conductor consisted of silver-plated constantan strip about 0·008 in. by 0·00075 in. Sixty junctions had a resistance of 85 ohms.

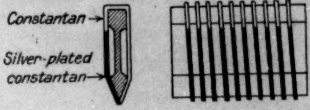


FIG. 9.

Mica was found more convenient than ebonite for many purposes, but it was soon apparent that the presence of any insulating material near the hot junctions materially lowered their temperature, besides rendering their action very much more sluggish. The effect of this is well seen in the photographic records in Fig. 10, for which I am indebted to Prof. A. V. Hill, F.R.S., and Mr. W. Hartree, of the Physiological Laboratory, Cambridge.

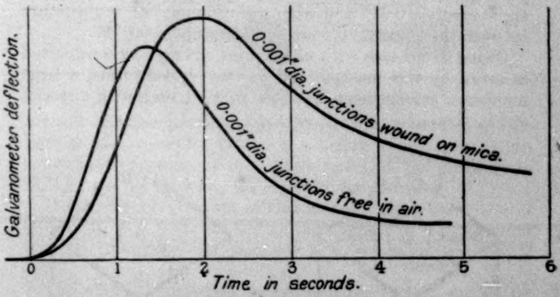


FIG. 10

The arrangement for mounting the junctions shown in Fig. 11 was therefore adopted, whereby the proximity of insulating material to the junctions is avoided. This arrangement is the same as that adopted by Prof. Hill in some of his work, in which he used junctions constructed in the usual manner ("Journal of Physiology," Vol. XLIII., No. 6, February, 1912).

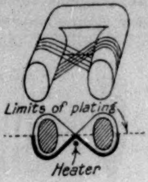


Fig. 11.

In order to deal with higher temperatures than could safely be used with silver and constantan, junctions were constructed by nickel-plating wires of nickel-chrome alloy specific resistance 575 ohms per circular mill foot, 0·032 in. diameter before plating and 0·0445 in. after plating. These junctions gave approximately 0·016 volt when the difference of temperature between the hot and cold junctions was about 900°F.

Owing to the ease with which junctions may be constructed in series by this method, and in very varied forms, a large number of arrangements has been devised, each with a special end in view.

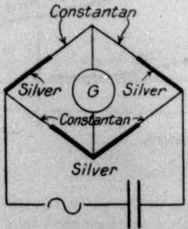


Fig. 12.

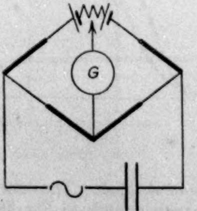


Fig. 13.

One type is in the form of a Wheatstone bridge arrangement (Fig. 12), the arms of which are balanced for resistance, but have thermo-junctions so arranged that the thermo-E.M.F.s cancel in the external or supply circuit, but assist one another in sending current through the measuring instrument in the branch circuit.* This arrangement utilises the high resistance portion of each junction as a heater, and is particularly useful in measuring high frequency alternating currents, since the inductance and resistance of the arms may be properly balanced so that none of the high-frequency current flows through the instrument, and the arrangement can be accurately calibrated on continuous current owing to the fact that the differential action of the thermo-junctions tends to maintain the sum of the thermo-E.M.F.s constant even though the alternating current may not divide between the two paths in precisely the same ratio as the continuous calibrating current. The error would be less than 1¼ per cent. even with 25 per cent. more current in one branch of the bridge than in the other.

For dealing with large currents it is convenient to have a series of thermo-junctions in each arm of the bridge. These may be constructed as shown in Fig. 14; the junctions here are made with much larger cross-sections and radiating surfaces at the cold junctions, so that the heat generated by the passage of the current results in a higher temperature at the constricted sections, and thus a series of preponderating E.M.F.s are set up along the strips in one direction.

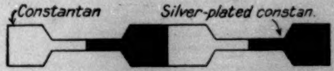


FIG. 14

One bridge constructed in this manner had a resistance of 0·37 ohm, and with an alternating current of 6·75 amperes through the bridge the thermo-E.M.F. generated was 0·0192 volt.

The above arrangement may be made more sensitive by inserting two equal sources of E.M.F. as shown in Fig. 13, and readjusting the balance of the bridge to compensate for the thermo-E.M.F.s set up by the circulating current generated by these sources of E.M.F. round the bridge circuit.

In certain respects these Wheatstone bridge arrangements are similar to the bolometric arrangement used in 1890 by

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* Since writing this Paper, we find that a similar arrangement, using the ordinary type of couples, has been described in 'British Patent No. 5413 of 1909.

H. Rubens and R. Ritter (" Weid. Annalen," Vol. XL., p. 56), while if the thermo-E.M.F.s are produced by the absorption of radiant heat instead of by a heating current they resemble Langley's bolometer. The chief difference from those arrangements is that the galvanometer is deflected by the thermo-E.M.F.s set up in each arm of the bridge instead of by a change in resistance of one arm.

Another useful arrangement consists of two lines of junctions connected in opposition and arranged close together as shown in Fig. 15. If radiant heat be arranged to fall in a line 1 mm. in width symmetrically about the axis A-B, the thermo-E.M.F.s generated in the two halves will be equal and opposite, but a movement of the heat line of ½ mm. to either side will cause it to cover entirely one set of junctions or the other, resulting in a deflection of the galvanometer from one side of zero to the other side of zero. Since the number of junctions in each line may be made large by these methods, the deflection of the galvanometer *G* may be made substantially proportional to the movement of the band of radiant heat.

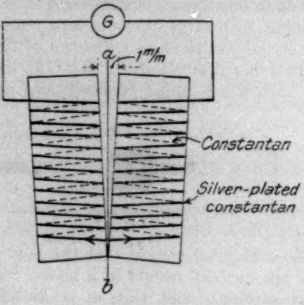


FIG. 15.

The arrangement, therefore, comprises a useful means of magnifying small movements. The band of radiant heat may be the image formed by the mirror of another reflecting galvanometer, hence it is possible to cause a deflection produced by this galvanometer to produce a considerably larger deflection in the galvanometer *G* connected in the thermo-electric circuit.

For such work as spectrum analysis the method enables lines of junctions in series to be constructed with ease and accuracy, which present a very small width to the radiation to be analysed, and should prove valuable for such purposes.

It is necessary to guard against the presence of insulating material in proximity to the working line of junctions as heat is absorbed by such material, and acts upon the junctions so that the full benefits are not obtained which would arise from the narrowness of the line of junctions.

For work where couples of exceedingly small mass are required, the methods offer great possibilities, since the only limitation to the fineness of the junctions is that imposed by mechanical questions of producing the wire. In this way it has been found possible to construct junctions with a diameter of less than 0·0002 in., and there seems no reason why this linit should not be exceeded.

Such junctions, owing to their very small mass, respond with great rapidity to changes of temperature.

For some purposes it is convenient to have a very large number of junctions connected in series, and occupying the smallest possible space. This may be readily done by forming lines of junctions on slips of mica, and assembling them with mica or paper between. In this way a pile has been constructed having 4,000 couples in series with a resistance of about 8,000 ohms, the hot junctions presenting an area of 1½ in. by ⅝ in.

ABSTRACT.

The method, which was devised to overcome the difficulty of making satisfactory soldered joints between the elements of thermopiles having a large number of closely packed junctions, consists in using a continuous wire of one of the elements and coating those parts of it which have to form the other element with an electrolytic deposit of another metal. If the conductivity of the latter is considerably greater than of the former, and a fairly thick sheath is deposited, a thermo-couple is produced which is not appreciably impaired in efficiency by the short-circuiting effect of the core. Constantan wires coated with either copper or silver sheaths were found to be suitable for most purposes.

DISCUSSION.

Mr. R. S. WHIPPLE said there were many cases in which the device would be of the greatest service to physicists, in the measurement of radiation, for instance. He might mention that Prof. Hill, of Cambridge, has recently been measuring the rise of temperature of nerves in action by means of these couples.

Mr. F. E. SMITH said that he had had to make a number of thermopiles at one time and another, and he fully appreciated the advantage of Mr. Wilson's method. It would be a great labour-saving device.

Mr. C. R. DARLING asked if the method would enable a radiation pyrometer similar to the Féry type to be made for the measurement of low surface temperatures.

Dr. J. S. **Anderson** emphasised the utility of the device for spectro-radiation work, as the total breadth could be made very small and a large number of junctions per centimetre employed. He had had the opportunity of using some of Mr. Wilson's couples, and was very pleased with them.

Dr. **Rayner** asked what difference of temperature could be detected. He thought the measurements should be given in the metric system to render the Paper comprehensible to foreigners.

Dr. **Vincent** asked if Wollaston wire had been tried. Had the winding to be done by hand, or could it be done on a lathe?

Prof. **Fortescue** suggested that the arrangement of Fig. 14 could be adapted to the measurement of quite high-frequency currents.

Dr. L. **Hopwood** asked if the Authors were certain of the effect which was utilised in the arrangement of Fig. 14. Benedicks had shown that if there was a steep temperature gradient, as would be the case in this arrangement, a reversed Thomson effect, comparable in magnitude to the Seebeck effect, was obtained. It would be interesting, therefore, to try using only one metal instead of two.

Mr. **Wilson** said there would be no difficulty in measuring low surface-temperatures with an instrument employing these junctions. As regards measurable temperature difference, he had not gone beyond about 1/1,000°C. He had not tried Wollaston wire, but he found that a patent had recently been applied for using such wire and dissolving off the sheath from one "element." Many of the couples shown were wound on lathes. In other cases the grooves were lathe-cut and the wire put on by hand, which was almost as quick. He had not made experiments with one metal with the arrangement of Fig. 14 as suggested by Dr. Hopwood, but would do so at the first opportunity. His results, however, appeared to be due to ordinary thermo-electric effects.

XXX. *On the Use of "Vacuum Arcs" for Interferometry.* By J. Guild, *A.R.C.Sc.*, *D.I.C.*, *F.R.A.S.* (From the National Physical Laboratory.)

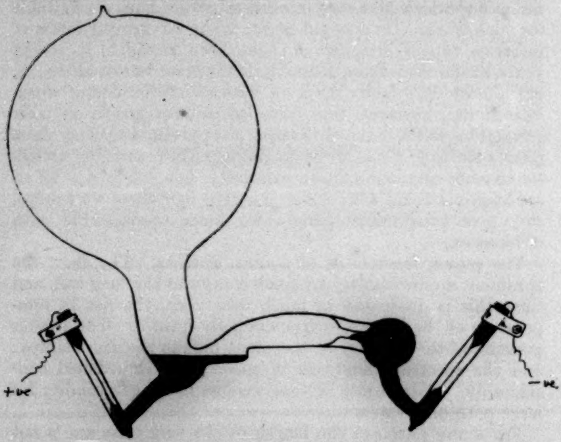
Received June 5, 1920.

In the use of interferometers of various types it is frequently necessary to obtain interference fringes of considerable path difference—for example, in testing the plane parallelism of thick slabs of optical glass. This necessitates the use of light which is monochromatic to a very high degree; and only radiations which give very narrow spectrum lines are suitable for such work. On account of the many convenient forms of mercury vapour lamp which have been available in recent years, the mercury lines, particularly the green line at 5461A.U., are almost universally used for general interferometer work. It is found, however, that such lamps vary greatly in their suitability if the path difference is appreciable. The short quartz lamps of high intrinsic brilliancy are practically useless for any but quite thin films; whereas the long lamps, with an arc length of about 4 ft., designed for the lighting of workshops, &c., give satisfactory fringes for quite considerable path differences.

The reason for this is, of course, obvious. The short arc consumes approximately as much energy as the long one, and since this is dissipated in much less space, the rise in temperature of the lamp is correspondingly greater. The vapour pressure of the mercury is, therefore, high in the short lamps; and the spectrum lines are in consequence broadened considerably. This results in a serious loss of fringe visibility for large path differences.

For many purposes the length of the long tube arc is not particularly inconvenient. Moreover, it is sometimes an advantage, as by the aid of mirrors and lenses one lamp may be made to illuminate several interference apparatus distributed around a room. For other purposes, however, its size and non-portability are objectionable, as it may be difficult to arrange it in the proper relationship to other parts of the apparatus. Not only so, but it is frequently desirable to have a source of greater brilliancy, but giving at the same time the narrow lines of the long arc. This can be accomplished quite simply. The vapour pressure in an enclosure is determined by the temperature of the coolest part. Even if the rate of evaporation in the hot part is very rapid, no great excess of pressure can exist unless there is a narrow constriction between the hot and cold regions.

The pressure in the short arc can, therefore, be considerably reduced by attaching a condensing chamber to the lamp. This has been done in a lamp designed by the author, and made by the Westinghouse Cooper Hewitt Co., Ltd. It consists of their ordinary laboratory quartz lamp, with the addition of a large bulb above the positive electrode which serves as a condensing chamber to keep the pressure low.



The diameter of the bulb is about 5 inches and the general proportions of the lamp are shown in the sketch.

The performance of this lamp is very satisfactory. When taking a current of 2·5 to 3 amperes, it produced visible fringes on an interferometer with as great a path difference as the four-foot arc, and the illumination of the field was very much brighter. With a current of 5 amperes, which is as much as this type of lamp should be run at for regular use, there is a very slight deterioration in the fringe visibility with long paths, but this is not sufficient to matter in practice.

It may be of interest to give some comparative figures of the performance of four mercury arcs of different types. The figures given are the maximum path differences, in air, for which fringes were observable with a particular interferometer arrangement.*

Arons-Lummer Water-Cooled Quartz Arc, in which the arc is in the form of a horse-shoe, enclosed in an outer cover, also of quartz, which is water-cooled.	4 cm.
Westinghouse Cooper Hewitt Quartz Lamp, type Y_1 ..	2 cm.†
Similar Lamp with Condenser, as described above	10 cm.
Westinghouse Cooper Hewitt Lamp, type K (4-*ft. arc*) .	10 cm.

It is clear from these figures that the first two lamps, representative of the ordinary patterns of short laboratory arc, are of very limited utility in general interference work as compared with either the third or fourth. Type three, by combining the high intrinsic brightness of the short arc‡ with the greater spectral homogeneity of the long arc, will be found of considerable utility by many workers.

It has not been possible to try this expedient with the Cadmium arc. Unfortunately the Cadmium lines, from their simple structure the most suitable of all for interference work, are not readily obtained of satisfactory brightness and homogeneity at the same time. For purposes where extreme homogeneity is not required, *e.g.*, polarimetry, the enclosed Cadmium vacuum arc of Dr. Sands§ gives a beautifully intense Cadmium spectrum; but owing to the high temperature which the lamp attains the lines are ruined for interference work, except with thin films.

There would probably be a difficulty in employing a condensing bulb to lower the pressure in a Cadmium arc, as the condensed metal might stick to the bulb instead of falling back into the arc. If this difficulty could be overcome, say, by choosing the bulb of such size that the Cadmium vapour was liquefied but not solidified, the use of these lines for interferometry might be greatly facilitated.

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* The visibility depends on several factors besides the homogeneity of the light; the figures are, therefore, comparative only.

† After running for some minutes so as to obtain a steady state.

‡ The intrinsic brilliancy of the quartz arc with condenser is distinctly less than that of the ordinary pattern for the same power consumption, but is much greater than that of the long arc.

§ Proc. Phys. Soc., Vol. XXVIII., p. 94.

ABSTRACT.

The Paper discusses the relative merits of short and long mercury arcs for this work, and points out that the defect of the former is due to the broadening of the spectrum lines consequent on the high vapour pressure within the lamp. It is shown that by attaching a condensing bulb to the lamp, so as to prevent excessive rise of vapour pressure, the short lamp can be made practically as good as the long one as regards sharpness of lines, while still being of much greater intrinsic brightness.

XXXI. *The Maintenance of a Vibrating System by Means of a Triode Valve.* *By* S. BUTTERWORTH, *M.Sc.* (*From the National Physical Laboratory.*)

RECEIVED MAY 27, 1920.

1. THE triode valve has been applied recently by Eccles[*] for the purpose of maintaining the vibrations of a tuning fork. The circuit employed is shown in Fig. 1. *V* is the triode valve, whose filament *F* is heated by the battery *FB*. The anode *A* is connected through the anode battery *AB*, and the coil *AC* to one terminal of the filament. The grid *G* is connected to the filament through the coil *GC*. The tuning fork *T* is permanently magnetised by an auxiliary magnet, on whose pole-pieces the coils *AC* and *GC* are wound, the poles of the fork being indicated by *n* and *s*.

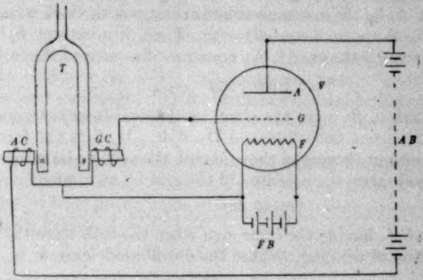

FIG. 1.

The action is as follows: Suppose the fork to have been set in vibration, and consider the moment when the two prongs are moving away from their respective coils. The motion of the pole *s* will induce an E.M.F. in the coil *GC*, which will raise the potential of the grid, and so increase the current flowing into the valve at the anode. This current leaving by the filament completes its circuit via the coil *AC*, round which it passes in such a direction as to repel the pole *n*, and thus assist its motion.

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[*] Eccles, Proc. Phys. Soc., Vol. XXXI., p. 269, 1919.

The same argument applies to any phase of the motion so that the vibrations are maintained.

The motion is only assisted if the relative directions of winding in the two coils are as in the figure. If the ends of the coils be disconnected from the anode battery and the grid respectively, and the open ends connected to a source of alternating current, the force exerted by the current flowing through the grid coil would oppose that due to the current in the anode coil if the winding is correct.

2. It is also necessary that the energy supplied by the valve shall be sufficient to balance the energy dissipated by the motion of the fork. If the fork be treated as a system of one degree of freedom, this condition may readily be determined.

Let A_1 be the mechanical force acting on the fork when unit current flows in the anode coil. Then, if a current I_a leaves the valve by the anode, the equation of motion of the fork is

$$(aD^2+\beta D+\gamma)y=A_1I_a \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

in which a, β, γ are the usual mechanical constants, y is the instantaneous deflection and $D=d/dt$. If A_2 is the force per unit current flowing in the grid coil, the voltage induced in the grid coil raises the potential of the grid by an amount

$$v=A_2Dy \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

A_1 and A_2 having the same sign when the coils have the same direction of winding, so that the extra anode current is

$$i_a=-gv=-gA_2Dy \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

in which g is the slope of the grid voltage — anode current characteristic of the valve at its working point, g being of the dimensions of a conductance and positive in sign.*

Since the normal anode current merely produces a steady deflection, we may, for the purpose of determining the motion, identify i_a with I_a, so that (1) becomes

$$\{aD^2+(\beta+A_1A_2g)D+\gamma\}\,y=0. \quad . \; . \; . \; . \; . \; . \quad (4)$$

The oscillations will increase, remain steady or decay, according as $\beta+A_1A_2g$ is negative, zero or positive.

Thus, for the maintenance of oscillations, A_1 and A_2 must be of opposite signs, and

$$g>-\beta/A_1A_2=g_0(\text{say}). \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

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* The effect of variations of anode voltage upon anode current is discussed in section 18.

The conclusion in regard to sign is identical with that arrived at in section I.

3. The quantity β/A_1A_2 is of the dimensions of a conductance, and it is important to form some notion of its magnitude. Suppose the anode and grid coils to have a mean radius of 1 cm., and a square channel section of 1 cm. side; and let them be wound with 10,000 turns of fine wire. The strength of the poles on the fork will be taken as 200 C.G.S. units. Then, assuming the turns concentrated in a single ring of 1 cm. radius, with the pole as centre, the force urging the pole when unit current flows is 4π megadynes. This gives the order of A_1 or A_2 if we may treat the fork as if its inertia were concentrated at the poles. Now, Rayleigh* shows that this may be done for the fundamental period, and that the equivalent inertia of one prong is then one-quarter its whole mass, the remainder of the prong acting as a spring devoid of inertia. This gives $a \doteq \frac{1}{2}$ (mass of one prong), since both prongs are vibrating. With a fork of reasonable dimensions, the mass of one prong may be taken as of the order 20 grams, and then $a=10$ grams.

β can be found from the rate of decay of the free oscillations and the value of a. Thus, suppose the oscillations to fall to $1/e$th of their initial value in one second; then $\beta=2a=20$.

Hence, taking $A_1=A_2=10^7$,

$$A_1A_2/\beta=5\times10^{12} \text{ C.G.S. units}=5{,}000 \text{ ohms},$$

since it is of the dimensions of a resistance.

The condition for maintenance of the above system is, therefore, that 1 volt on the grid should develop an anode current of at least 200 microamperes.

A coil of the above dimensions and turns wound with No. 46 wire would have a direct current resistance of about 4,000 ohms, and its inductance with an air core would be about 4 henries.

For other turns and the same overall dimensions, A_1, A_2, the resistance and the inductance all increase as the square of the turns.

4. It has been suggested† that a condenser placed in parallel across the anode or grid coil would render the oscillations easier to maintain, and this has proved of value in certain cases. That a condenser might improve matters is to be ex-

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[*] "Theory of Sound," Vol. I., p. 289.

[†] Proc. Phys. Soc. (*loc. cit*). Discussion.

pected from the principles of resonance, although when a considerable current is flowing in the condenser-coil circuit there is an additional energy loss due to the resistance of the coil. Again, the optimum capacity will not necessarily be that which will make the free frequency of the electrical circuit equal to that of the fork, as the reactive effect due to the motion of the fork is by no means negligible.

5. In the following theory use is made of a principle of electrical equivalence developed by the author* in an earlier Paper, in which it was shown that a dynamical system kept in motion by the magnetic action of a current flowing in a coil could be replaced by an electrical system in series with the coil, and of such a nature as to produce the same back E.M.F. as that produced by the motion of the dynamical system.

For the case of a system of one degree of freedom, the equivalent electrical system consists of a parallel combination of inductance, capacity and conductance. In the present problem the only difference is that two coils are acting instead of one. The necessary modifications on the earlier theory are easily made.

6. If the two coils carry currents, I_1 and I_2, they act on the dynamical system with forces A_1I_1 and A_2I_2. The equation of motion is

$$(\alpha D^2+\beta D+\gamma)y=A_1I_1+A_2I_2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the back E.M.F. on the respective coils are

$$E_1=A_1Dy, \quad E_2=A_2Dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

so that, eliminating y by means of (6),

$$E_1=I_1/\xi_1+I_2/\xi_{12}, \quad E_2=I_1/\xi_{12}+I_2/\xi_2 \quad . \quad . \quad . \quad (8)$$

in which

$$A_1^2\xi_1=A_1A_2\xi_{12}=A_2^2\xi_2=\alpha D+\beta+\gamma/D \quad . \quad . \quad . \quad (9)$$

Equations identical with (8) will be obtained if, instead of the dynamical system, we place in series with the respective coils inductances (without resistance) of values A_1^2/γ, A_2^2/γ, perfectly coupled with each other, and shunt the former by means of a leaky condenser of capacity α/A_1^2 and leakage conductance β/A_1^2. The leaky condenser may be transferred if desired to the second coil, the values of capacity and leakance being then α/A_2^2 and β/A_2^2. This follows from the properties of a perfectly coupled transformer.

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* Proc. Phys. Soc., Vol. XXVII., p. 410, 1915.

7. Applying this principle to the case of the maintained fork, the system may now be taken as purely electrical, the circuits being shown in Fig. 2. l_1, l_2 are the equivalent inductances of values A_1^2/γ, A_2^2/γ perfectly coupled. c_1 is the equivalent condenser a/A_1^2, and s_1 its leakage conductance β/A_1^2. L_1 is the inductance and R_1 the resistance of the anode coil, L_2 R_2 the inductance and resistance of the grid coil. The condensers whose effects are to be investigated are C_1 and C_2.

The action of the equivalent leaky condenser may be taken into account by first obtaining the equations which hold when the condenser is absent, and then replacing the resistance operators l_1D, l_2D and the mutual operator $\sqrt{l_1l_2}D$ by $1/\xi_1$, $1/\xi_2$, $1/\xi_{12}$, the values of ξ_1, ξ_2, ξ_{12} being given by (9).

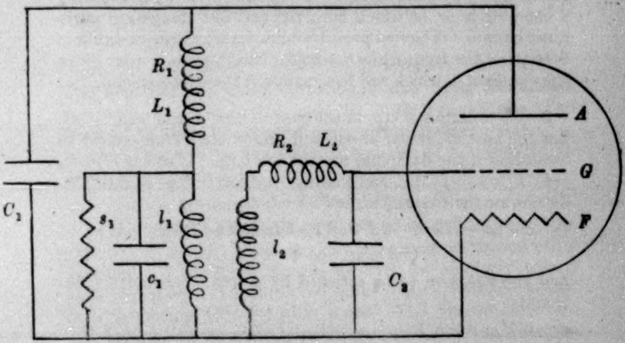

FIG. 2.

Let i_1, i_2 be the clockwise currents* circulating in the anode and grid circuits respectively. The grid voltage is i_2/C_2D, and the resulting anode current leaving the valve is $-gi_2/C_2D$. This current flows back to the filament via C_1, thus impressing an E.M.F. on the anode circuit of amount $-gi_2/C_1C_2D^2$. Hence, if X_1, X_2 are the resistance operators of the anode and grid circuits, and m is the mutual operator,

$$\left.\begin{aligned} X_1i_1&=(m-g/C_1C_2D^2)i_2 \\ X_2i_2&=mi_1 \end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

---

* The currents and voltages referred to are the deviations of the currents and voltages from the normal values when oscillations are excited.

from which

$$(X_1X_2/m-m+g/C_1C_2D^2)\{i_1, i_2\}=0 \quad . \quad . \quad . \quad . \quad (11)$$

Using the explicit values of X_1, X_2, m, viz.,

$$X_1=R_1+L_1D+1/C_1D+1/\xi_1,\ X_2=\mathrm{R}_2+L_2D+1/C_2D \\ +1/\xi_2,\ m=1/\xi_{12},$$

(11) becomes, with the help of (9),

$$\{(aD^2+\beta D+\gamma)(C_1L_1D^2+C_1R_1D+1)(C_2L_2D^2+C_2R_2D+1) \\ +A_1{}^2C_1D^2(C_2L_2D^2+C_2R_2D+1)+A_2{}^2C_2D^2(C_1L_1D^2 \\ +C_1R_1D+1) \\ +A_1A_2gD\}\ \{i_1, i_2\}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The three possible oscillation frequencies indicated by (12) are obtained by putting $D^2=-\omega^2$ in the operator, thus giving a complex in ω, of which both the real and imaginary parts must be zero. The real part is a cubic in ω^2, whose three roots determine the frequencies $(\omega/2\pi)$. The imaginary part gives the values of g, which will just maintain these frequencies.

8. The solution of (12) in the general case would be difficult, but the case of similar anode and grid circuits is amenable to treatment if the dampings are not too large. Putting $C_1=C_2=C$, $L_1=L_2=L$, &c., and writing $-\omega^2$ for D^2, the equation to determine the frequencies is

$$(\gamma-a\omega^2)(1-CL\omega^2)^2-2A^2C\omega^2(1-CL\omega^2)(1+\beta R/A^2) \\ -C^2R^2\omega^2(\gamma-a\omega^2)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and the equation to be satisfied by g (the conditional equation) is

$$\beta(1-CL\omega^2)^2+2CR\{(\gamma-a\omega^2)(1-CL\omega^2)-A^2C\omega^2(1+\beta R/2A^2)\} \\ \pm A^2g=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The positive sign is used when the coils are similarly wound and the negative sign when oppositely wound.

Putting further,

$$\gamma=a\omega_1{}^2,\ 1/CL=\omega_2{}^2,\ R/L=\mu\omega_1,\ \beta/a=\nu\omega_1 2(A^2/La)\,(1+\beta R/A^2) \\ =\sigma^2\omega_1{}^2,$$

(13) and (14) become

$$(\omega_1{}^2-\omega^2)(\omega_2{}^2-\omega^2)^2-\sigma^2\omega^2\omega_1{}^2(\omega_2{}^2-\omega^2) \\ -\mu^2\omega^2\omega_1{}^2(\omega_1{}^2-\omega^2)=0 \quad . \quad . \quad . \quad . \quad (15)$$

$$\nu\{(\omega_2{}^2-\omega^2)^2+\mu^2\omega^2\omega_1{}^2\}+\mu\{2\omega_1{}^2-\omega^2)(\omega_2{}^2-\omega^2)-\sigma^2\omega^2\omega_1{}^2\} \\ \pm A^2g\omega_2{}^4/\omega_1 a=0 \quad . \quad . \quad . \quad . \quad . \quad (16)$$

9. In the case when the free frequency of the circuits is equal to that of the fork (that is, $\omega_2=\omega_1$), the frequency equation immediately factorises, giving

$$\omega^2/\omega_1^2=1,\ 1+\tfrac{1}{2}(\sigma^2+\mu^2)\pm[\{1+\tfrac{1}{2}(\sigma^2+\mu^2)\}^2-1]^{\frac{1}{2}}.$$

Thus, the three possible oscillation frequencies are respectively below, equal to, and above the free frequency of the fork. If, as is often the case, σ and μ are small compared with unity,

$$\omega/\omega_1\doteqdot 1,\ 1\pm\tfrac{1}{2}(\sigma^2+\mu^2)^{\frac{1}{2}}.$$

For the root $\omega=\omega_1$, the conditional equation reduces to

$$\mu(\nu\mu-\sigma^2)\pm A^2g/\omega_1 a=0.$$

Now, $\sigma^2>\nu\mu$, so that for the above condition to be possible the coils must be *similarly* wound. Inserting the values of σ, μ, ν, and using the positive sign,

$$g=2R(1+\beta R/2A^2)/L^2\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For the other roots, the conditional equation reduces to

$$(\nu+\mu)(\sigma^2+2\mu^2)\omega^2\pm A^2g\omega_1/a=0.$$

The negative sign must be used, so that these oscillations can only occur when the coils are *oppositely* wound. The *lower* of the two frequencies requires the smaller value of g, so that this will be the actual frequency of the oscillation. When σ and μ are small, $\omega\doteqdot\omega_1$, and the condition is then

$$g=(\nu+\mu)(\sigma^2+2\mu^2)a\omega_1/A^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the typical example of section 3, $\sigma^2=0{\cdot}00025$, $\mu^2=0{\cdot}025$, $\nu^2=10^{-7}$, a free fork frequency of 1,000 cycles per second being assumed. With these values, we find for similarly wound coils (equation 17) $1/g=50{,}000$ ohms ; for oppositely wound coils (equation 18), $1/g=200$ ohms. The former condition corresponds to 20 microamperes per volt, and the latter to 5 milliamperes per volt. The oscillation with similarly wound coils is, therefore, by far the easiest to maintain. The value of g for the oppositely wound coils will, however, rapidly diminish with increase of L/R, because of the factor μ^3. Also, as will be shown in the next section, a small variation of ω_2 (by capacity adjustment) will cause a gain in ease of maintenance.

10. If the frequency of the electrical circuits is varied over a short range in the neighbourhood of $\omega_1/2\pi$, we may write in (15) and (16)

$$\omega_2^2-\omega^2=x\omega_1^2,\ \omega_1^2-\omega^2=y\omega_1^2\,;$$

and in the case σ^2, μ^2 small, they reduce to

$$x(xy-\sigma^2)-\mu^2 y=0$$

$$\nu(x^2+\mu^2)+\mu(2xy-\sigma^2)\pm A^2 g/\omega_1 a=0\ ;$$

or, expressing g and y in terms of x

$$y=\sigma^2 x/(x^2-\mu^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

$$\pm g=-(a\omega_1/A^2)(x^2+\mu^2)\{\nu x^2+\mu(\sigma^2-\mu\nu)\}/(x^2-\mu^2). \quad . \qquad (20)$$

Since $\sigma^2>\mu\nu$, the oscillations which occur when the coils are similarly wound are such that $x^2>\mu^2$. In this case g is least when $x=0$, so that the value of g given in (17) is the minimum g for similarly wound coils.

The oscillations which occur when the coils are oppositely wound are such that $x^2>\mu^2$. A minimum value occurs for g when

$$x^2-\mu^2=\mu\sigma\sqrt{2\mu/\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

and then

$$g=(a\omega_1/A^2)(\mu\sqrt{2\nu}+\sigma\sqrt{\mu})^2, \quad . \quad . \quad . \qquad (22)$$

while

$$y=x\sigma\sqrt{\nu/2\mu^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

In the typical case at minimum g

$$x\doteqdot 0{\cdot}32,\ y\doteqdot 0{\cdot}001,\ 1/g\doteqdot 15{,}000 \text{ ohms.}$$

11. The value of x in the typical case is thus hardly within the approximation assumed in obtaining (20); y is, however, small.

If we do not assume x small, equation (20) must be modified by replacing g in that equation by $g(1+x)^2$, y being still supposed small. We have thus really found when $g(1+x)^2$ is a minimum. The true minimum g will occur for a larger x when x is positive, and the value of g will be less than that calculated above. When x is negative, the minimum will occur for a smaller value of x, and the value of g will be larger than that for the positive x. Thus, of the two possible minima, the best is that for which $\omega_2>\omega_1$.

12. The full behaviour of the system in the neighbourhood of $\omega_2=\omega_1$ may be exhibited graphically for given values of σ, μ, ν, $a\omega_1/A^2$.

From equation (19) y is calculated for a suitable series of values of x, and $-y$ is plotted against $x-y$. The ordinates then give $(\omega^2-\omega_1^2)/\omega_1^2$, and the abscissæ $(\omega_2^2-\omega_1^2)/\omega_1^2$. The graph, therefore, gives the variation of the oscillation frequency from the free frequency of the fork as the free electrical frequency is varied by capacity adjustment.

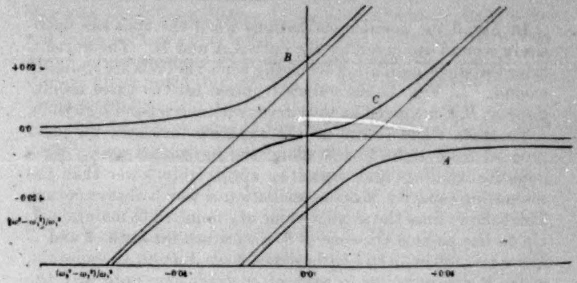

FIG. 3.

Similarly, from the (modified) equation (20), g is calculated and plotted against $x-y$. This graph shows the condition for maintenance of the oscillations given in the former graph.

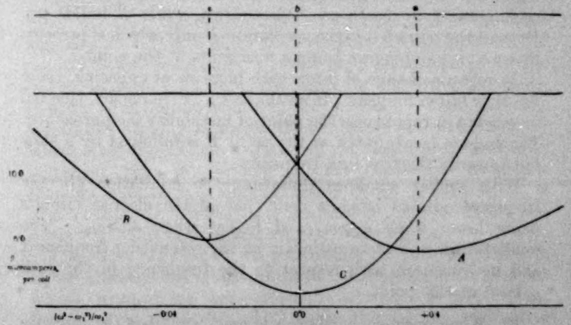

FIG. 3A.

$\omega_1/2\pi$ = free frequency of fork. $\omega_2/2\pi$ = free frequency of circuits. $\omega/2\pi$ = frequency of maintained oscillations.

The case used for Figs. 3 and 3A is

$$\sigma=0{\cdot}012,\ \mu=0{\cdot}02,\ \nu=0{\cdot}002,\ A^2/a\omega_1=2 \text{ ohms},$$

which corresponds, for example, to

$$\omega_1=5{,}000,\quad R=1{,}250 \text{ ohms},\quad L=12\cdot5 \text{ henry},\quad \beta/\alpha=10,$$
$$\alpha/A^2=100 \text{ microfarads}.$$

In Fig. 3 the possible oscillations when the coils are oppositely wound are given by the curves A and B. The curve C is for the single oscillation occurring when the coils are similarly wound. In Fig. 3A the values required for the three oscillations *A*, *B*, *C* are given by the curves with corresponding letters.

To trace what occurs as the capacity increases we must proceed from right to left along the horizontal axis. With opposite windings and capacities appreciably lower than the resonating capacity, it is the oscillation *A* which always occurs. This follows from the smaller value of g required to maintain *A*. Up to the point a the ease of maintenance for both *A* and *B* increases. From a to b, *A* becomes more difficult to maintain, while *B* still increases in ease of maintenance until at b the ease of maintenance of either oscillation is the same. A slight increase of capacity at the point b will make *B* easier to maintain than *A*, so that there will be a sudden increase in frequency from that corresponding to *A* to that corresponding to *B*. If the frequency difference is suitable, this will be accompanied by beats as *A* dies down, while *B* builds up. Beyond the point b it is the oscillation *B* only which is present, its ease of maintenance being a maximum at the point c.

As regards change of pitch with increase of capacity, there is a slow fall in frequency from the free fork frequency, this fall increasing in rapidity as the point of instability is approached. The sudden rise in pitch at the point *B* is followed by a slow fall towards the free fork frequency.

With similar windings (following the oscillation *C*), the frequency always remains near that of the electric circuits being lower when $\omega_2>\omega_1$ and higher when $\omega_1>\omega_2$. The oscillation is easiest to maintain at the resonating frequency, and its variation with respect to the frequency of the fork is least at this point.

13. When only one condenser is used, equation (12) reduces to

$$\{(\alpha D^2+\beta D+\gamma)(CLD^2+CRD+1)+A_1{}^2CD^2+A_1A_2gD\}\{i_1,\ i_2\}=0, \quad . \quad . \quad . \quad . \quad (24)$$

in which A_1 refers to that coil across which the condenser is placed.

Putting $D^2=-\omega^2$, and using the notation of equation (15), the frequency equation is

$$(\omega_1{}^2-\omega^2)(\omega_2{}^2-\omega^2)=\tfrac{1}{2}\sigma^2\omega_1{}^2\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

The two possible oscillation frequencies, therefore, lie outside the range $\omega_1 \backsim \omega_2$, one being above and the other below this range. Since usually σ^2 is small, the roots (Ω_1, Ω_2) of (25) do not differ much from ω_1 and ω_2, their approximate values being given by

$$(\Omega_1-\omega_1)/\omega_1=(\omega_2-\Omega_2)/\omega_2=\sigma^2\omega_1{}^2/4(\omega_1{}^2-\omega_2{}^2), \quad . \qquad (26)$$

when ω_1 and ω_2 are not nearly equal, and by

$$\Omega_1-\omega_1=\omega_2-\Omega_2=\sigma\omega_1/2\sqrt{2} \quad . \quad . \quad . \quad . \qquad (26\text{A})$$

when $\omega_1=\omega_2$.

14. The conditional equation for Ω_1, Ω_2 is

$$\nu(\omega_2{}^2-\Omega^2)+\mu(\omega_1{}^2-\Omega^2)+A_1A_2\omega_2{}^2g/a\omega_1=0 \quad . \quad . \qquad (27)$$

This may be written

$$g=-\frac{a\omega_1}{A_1A_2\Omega^2}\;\frac{\mu(\omega_1{}^2-\Omega^2)^2+\frac{1}{2}\sigma^2\nu\omega_1{}^2\Omega^2}{\omega_1{}^2(1+\frac{1}{2}\sigma^2)-\Omega^2} \quad . \quad . \qquad (28)$$

by the substitution

$$\omega_2{}^2=\Omega^2\{\omega_1{}^2(1+\tfrac{1}{2}\sigma^2)-\Omega^2\}/(\omega_1{}^2-\Omega^2). \quad . \quad . \quad . \qquad (29)$$

Since (by (29)) the denominator of (28) has always the same sign as $\omega_1{}^2-\Omega^2$, and since the two roots are on either side of ω_1, it follows that A_1 and A_2 must have the same sign for the higher frequency and opposite signs for the lower frequency in order to make the condition (28) possible. Thus, with the winding of Fig. 1, it is always the oscillation of lower frequency which is set up. If the capacity of the condenser is gradually increased, thus diminishing ω_2, the oscillation frequency remains near to and slightly below $\omega_1/2\pi$ so long as ω_2 is greater than ω_1. When ω_2 is less than ω_1, the oscillation frequency remains near to and slightly below $\omega_2/2\pi$.

The effect may be shown graphically by plotting (25) with $\omega_2{}^2$ as abscissæ and ω^2 as ordinates. The result is a hyperbola whose asymptotes are $\omega^2=\omega_1{}^2$, $\omega^2=\omega_2{}^2+\frac{1}{2}\sigma^2\omega_1{}^2$. That branch of the hyberbola which passes through the origin gives the frequencies of the oscillations which may be set up when the coils are connected as in Fig. 1.

The comparative independence of the oscillation frequency of the value of the capacity used may account for the known independence of frequency of a maintained fork of such factors as variation of the steady anode voltage.

In electrically-excited oscillations the change in the anode battery will alter (by its change in earth capacity) the effective capacity across the anode circuit. This will have its full effect on frequency in electrically coupled circuits, but practically none in the case of the vibrating fork.

15. The condition for easiest maintenance may be obtained from (27) as follows :—

When Ω is nearly equal to ω_2, the term involving ν may be neglected and

$$g \doteqdot \frac{a}{A_1 A_2} \frac{R}{L} (1-\omega_1^2/\omega_2^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

For similar anode and grid coils, a/A_1A_2 is the equivalent capacity due to the reaction of the system on one coil, and is high (100 microfarads in the typical case). The oscillations are therefore difficult to maintain. When Ω is in the neighbourhood of ω_1, put $\omega_1^2-\Omega^2=v$ in (28), and retain only the first order term. Then g is of the form

$$g=K(v^2+m^2)/(v+n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

in which

$$K=-a\mu/A_1A_2\omega_1, \quad m^2=\tfrac{1}{2}\sigma^2\omega_1^4\nu/\mu, \quad n=\tfrac{1}{2}\sigma^2\omega_1^2.$$

It is a minimum in regard to variations in v when v satisfies the quadratic

$$v^2+2vn-m^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

and then

$$g=2Kv=g_1 \text{ (say)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

The electrical frequency required for this is found by substituting v for $\omega_1^2-\omega^2$ in (25). This gives

$$\omega^2=\omega_1^2(1+n/v). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

Equation (32) has two roots of opposite sign, of which the positive one has the smaller absolute value, and will therefore give the smallest possible value of g in (33). Since it is positive, K must also be positive, or the two coils must be oppositely wound. Further, from (31), since when no condenser is present, $v=0$,

$$Km^2/n=g_0=-\beta/A_1A_2 \text{ (by (5))},$$

as might otherwise be determined by substitution of the values of K, m, n. Using this in (33) to eliminate K,

$$g_1=2g_0vn/m^2=2g_0(v/n)(n/m)^2$$

Now (32) may be written

$$(v/n)^2+2(v/n)+m^2/n^2=0 \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

so that g_1/g_0 is a function of the single variable $(n/m)^2$. Denoting this by η and using the positive root of (35)

$$g_1=2g_0(\sqrt{\eta+\eta^2}-n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

while (34) becomes

$$\omega_2{}^2=\omega_1{}^2(1+\eta+\sqrt{\eta+\eta^2}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The oscillation frequency is most simply expressed by

$$\Omega_1{}^2=\omega_1{}^2(1+\tfrac{1}{2}g_1/g_0 \, . \, \beta L/aR). \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In terms of the fundamental constants

$$\eta=R(R+A^2/\beta)/L^2\omega_1{}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

It may thus be expressed in terms of two phase angles, θ and φ, φ being that of the coil alone and θ the modified phase angle obtained by adding the leakage resistance of the equivalent condenser of Fig. 2 to the resistance of the coil. In fact, θ is the actual phase angle of the coil when fed with currents of frequency $\omega_1/2\pi$, the reaction of the moving system being taken into account.

In terms of these phase angles

$$\eta=\cot\theta\cot\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39\text{A})$$

16. In the typical case (Sections 3 and 9) $\eta=0{\cdot}056$, so that $g_1=0{\cdot}38\ g_0$, $\omega_2=1{\cdot}14\ \omega_1$; and since $\beta L/aR=0{\cdot}002$, $\Omega_1=(1-0{\cdot}00019)\omega_1$. Since $1/g_0=5{,}000$ ohms, $1/g_1=13{,}000$ ohms, so that 1 volt on the grid must now be capable of developing 77 microamperes. The three cases treated may be summarised thus:—

Condensers used.	g min microamperes per volt.	Optimum electrical frequency.	Oscillating frequency.
0	200	...	1000·0
1	77	1 140	999·8
2	20	1 000	1000·0

The value of the added condensers is most marked when η is small. When this is so, (36) and (37) may be written

$$g_1=2g_0\sqrt{\eta}, \ \omega_2=\omega_1(1+\tfrac{1}{2}\sqrt{\eta}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

For higher values of η, the values of g_1/g_0 and $\omega_1{}^2/\omega_2{}^2$ may be exhibited graphically. This is done in Fig. 4 up to $\eta=2$. When $\eta>2$, the addition of a condenser is of little value.

17. If in (25) and (27) we make the substitutions

$$x=\omega_1{}^2/\omega_2{}^2,\quad y=\omega_1{}^2/\omega^2$$

we obtain
$$y=x/y=1+px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$
and
$$q(x-1)-x/y+1+z=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$
in which

$$p=1+\tfrac{1}{2}\sigma^2,\ q=aR/(\alpha R+\beta L),\ z=A_1A_2Lg\alpha/(\alpha R+\beta L).$$

Eliminating x/y between (41) and (42), the latter may be replaced by
$$y+z=(p-q)x+q \quad . \quad . \quad . \quad . \quad . \quad . \quad (42\text{A})$$

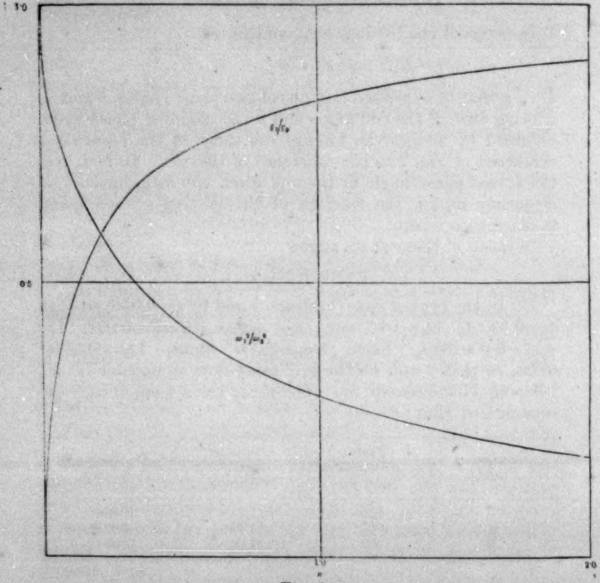

FIG. 4.

In these forms, the frequency equation (41) and the conditional equation (42A) admit of a simple graphical solution which holds for all values of the constants.

Taking axes x and y, draw the hyperbola (41) and on the same diagram draw the straight line $y=(p-q)x+q$. Then the ordinates of the hyperbola are proportional to the squares of the oscillation periods and the abscissæ to the capacity required to produce them. The vertical distances of the straight line from the hyperbola gives the values of z, being positive above and negative below the line. Since z is proportional to g, its smallness is a measure of the ease of maintenance of the corresponding oscillation. The upper branch of the hyperbola gives the periods when the coils are oppositely wound and the lower branch when they are similarly wound.

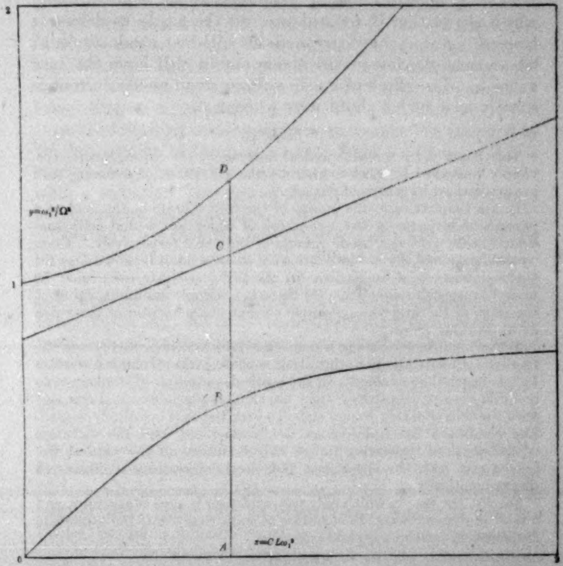

FIG. 5.

This construction is made in Fig. 5 for the hypothetical case $p = 1{\cdot}2$, $q = 0{\cdot}8$. If OA represents the capacity of the condenser, then the square of the two possible periods are represented by AB and AD, the former occurring when the coils are similarly wound. The values of g required are proportional to BC and CD respectively. The figure shows clearly the two optimum capacities and the advantage of using opposite windings.

18. In developing the theory, the effect of fluctuations of the anode voltage upon the anode current has been ignored. If v and v' be the variations in voltage of the grid and anode, the current leaving the valve by the anode varies by an amount

$$i_a=-gv-g'v'.$$

This relation may be written

$$i_a+g'v'=-gv,$$

which shows that if we still suppose the anode to deliver a current $-gv$, but shunt the anode circuits by a conductance g', the current flowing in the anode circuit will have the true value i_a. The effect of anode voltage upon anode current is exactly as if such a shunt were present.

ABSTRACT.

This Paper gives a mathematical analysis of the arrangement, previously described by Eccles, whereby the vibrations of a tuning fork are maintained by means of a triode.

In the simplest case, the prongs of the fork vibrate in the field of a permanent magnet, on the pole-pieces of which are wound coils connected in the grid and anode circuits of the valve respectively. From a consideration of the energy relations of the system, it is shown that for the maintenance of oscillations (a) the grid and anode coils must be wound in opposite directions, (b) there is a certain minimum value of the slope of the grid voltage-anode current characteristic of the valve at its working point.

Following this, the case in which condensers are connected across the two coils is treated on the principle of replacing the dynamical system by its electrical equivalent. In the particular example of similar anode and grid circuits, this system gives rise to three possible oscillations, one with the coils similarly wound and two with the coils oppositely wound. The conditions for maintenance are worked out, and the variation of the obtained frequency in the neighbourhood of the natural frequencies of both the dynamical and electrical systems is illustrated diagrammatically.

Finally, the case in which only one condenser is used is analysed, and a table is given showing the condition of maintenance and the oscillating frequency in the three typical examples.

DISCUSSION.

Dr. VINCENT said the question was one that would become of increasing importance. He would not be surprised to see instruments with valve-maintained vibrations taking the place of pianos.

Prof. FORTESCUE said the results were of importance apart from the application to tuning forks. In paragraph 5 the analogy between the electrical and mechanical system was enunciated; but later on, at the end of paragraph 14, they are said to behave differently under alterations in the voltage of the anode battery. He did not see how this should be. The calculations referred to a circuit of relatively low resistance. In practice, the pole-piece coils must be of high resistance. The wide range of tuning is probably due to this. Equation 12 and the results deduced from it were similar to those obtained by other workers, and it would be useful to have them all collected and put in the same notation.

XXXII. *Radiation and Convection from Heated Surfaces.* By

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I. OBJECTS OF EXPERIMENT.

THE question of the relative and absolute amounts of radiation and convection from heated surfaces of various shapes and sizes in a fluid—especially in air—is important from its intrinsic interest as well as from its commercial applications. The latter include hot wire ammeters, voltmeters, wattmeters, and anemometers,* psuchrainometers,† tube boilers and radiators.

In the following investigation "convection" is assumed to be that portion of the heat emitted from a surface, which is carried away by and in the direction of the cooling fluid; while "radiation" is emitted equally in all directions from a homogeneous heated surface.

Newton's law that the rate of cooling of a hot body placed in a current of air moving with uniform speed is proportional to its excess of temperature over that of the moving fluid has been shown experimentally to be approximately true in many cases, *e.g.*, for spheres heated in air to various temperatures, at various pressures,‡ and in currents at different speeds,§ and for metallic wires heated in air by an electric current.|| Experiments carried out about four years ago by one of us ¶ showed that in the case of wires of about 1 mm. in diameter raised 10°C. or 12°C. above the temperature of the air in a cylindrical enclosure surrounding them, radiation formed a comparatively small proportion of the total heat (2·5 to 4·5 per cent. if the wire was "bright"), most of the loss of heat being due to convection. Newton's law was found to be applicable within the limits of temperature employed. In those experiments and in the determinations described in the present Paper the "convection" was what

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[*] U. Bordini, "Nuovo Cimento," Pisa, 3, pp. 241-283, April 1912. J. T. Morris, "Electrician," Oct. 4, 1912. L. V. King, Roy. Soc. "Phil. Trans.," 373, Nov. 12, 1914. *See also* a Paper on "A Directional Hot-wire Anemometer," by Mr. J. S. G. Thomas, read before the Physical Society, March 12, 1920.

[†] J. R. Milne, Journ. Scott. Met. Soc., XVI., XXIX., 1912.

[‡] P. Compan, "Comptes Rend.," 133, p. 1813, 1901; and 134, p. 522, 1902.

[§] Crichton Mitchell, Trans. Roy. Soc., Edin., XLI., 4, p. 39, 1900.

[||] Kennelly and Sanborn, Amer. Phil. Soc. "Proc.," 55-77, 1914.

[¶] Proc. Phys. Soc., XXVIII., Dec. 1, 1915.

is usually termed "natural" or "free" convection, *i.e.*, that caused by the movement of the air due to the excess of temperature of the surface over that of the air surrounding it. The present experiments, however, differ from those of the former Paper in that the temperature difference is greater, the surfaces are larger, and they are not enclosed in an outer jacket. The surfaces tested are cylinders (including a flat surface) and spheres of different radii of curvature.

A mathematical basis for Newton's law in the case of convection of heat from a cylindrical surface by a fluid moving at right angles to the heated surface (*i.e.*, "forced" convection) has been given by A. Russell,* who also showed that the loss of heat per square centimetre must be inversely proportional to the square root of the diameter of the cylinder. The results of the experiments described below are in accord with the theory in the case of "natural" convection, *when convection alone is considered*, *i.e.*, when the heat due to radiation is subtracted from the total heat emitted. In the case of spheres the convection per unit area appears to be inversely proportional to the *cube root* of the diameter. (*See* Section V., Tables VIII. and IX.).

II. THEORY AND METHOD OF EXPERIMENT.

The temperature of the surface was raised to about 100°C. by passing steam through the hollow cylinder or sphere.† The total heat lost from the surface (convection + radiation) was obtained from the equivalent mass of steam condensed, (1) when the surface was "bright," (2) when it was covered with a "dead black" varnish.

The relative amounts of radiation from "bright" and "black" surfaces were then measured by a thermopile in conjunction with a low resistance galvanometer. This ratio was sensibly the same whether the surface was plane (as in a "Leslie" cube) or curved (as in a spherical surface). As regards convection it was assumed (as shown by Dulong and Petit in their classical research on emission of heat from various surfaces) that the convection from a particular surface is unchanged by coating it with dead black, the increase in emission being in that case all due to increased radiation.

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* A. Russell, Proc. Phys. Soc., XXII., p. 432, 1909.

† The method is somewhat similar to that employed by J. A. Hughes, as described in a Paper "On the Cooling of Cylinders in a Stream of Air," "Phil. Mag.," XXXI., pp. 118-130, Feb., 1916.

The exact temperature of each surface when steam was passing through the tubes was directly measured by means of a thermojunction of very thin copper and constantan wires (*see* Table I.). The junction was soldered into a "notch" made in the outside of the tube, the solder being then filed down to the same curvature as the rest of the surface. In the case of the glass spherical surface, the thermojunction was secured firmly in the notch by means of a thin strip of tin foil.

Now, assuming that the difference of the temperatures between the outside and the inside of the tube is given by $t_1-t_2=\frac{H.d}{K.A}$, where H is the heat emitted per second from area A of a tube of thickness d, we find that t_1-t_2 is of the order 0·006°C. It is by no means safe, however, to assume that the temperature of the inner surface of the copper is the same as that of the steam. Assumptions of the nature have, in fact, led to astonishing errors in some of the earlier determinations of coefficients of thermal conductivity of metallic conductors.

Let h_1, r_1, and c be the total heat, radiation, and convection respectively from the black surface per second, per square centimetre, per 1°C. excess temperature, and h_2, r_2, and c corresponding quantities for the bright surface, and let $h_1=ah_2$, i.e., $c+r_1=a(c+r_2)$. Further, let $r_1=br_2$. Then

$$\frac{r_2}{h_2}=\frac{a-1}{b-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{r_1}{h_1}=\frac{b(a-1)}{a(b-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) give the proportion of radiation to total heat lost for "bright" and black surfaces respectively; "a" being obtained from the total heat experiments and "b" from the "radiation" experiments. The actual amounts of "total heat," "radiation," and "convection" from each surface can thus be easily found.

III. Apparatus and Experimental Work.

The apparatus employed for measuring the "total heat" is sketched in Figs. 1 and 2.

The copper tube *AB* (Fig. 1) was suspended by two thin cords at a slope of about 1 in 7 from the horizontal, about

50 cm. from the ceiling of a small room free from draughts. (A few experiments were made with the tube vertical, the results being very slightly lower.) The dimensions of the room were 3·5×2·5×3·5 metres.

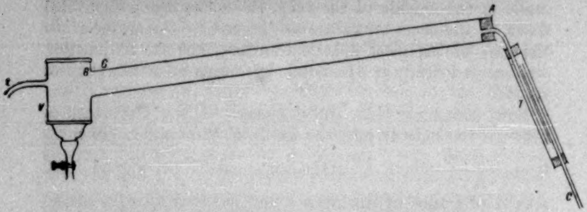

FIG. 1.—APPARATUS FOR MEASURING "TOTAL HEAT" FROM A CYLINDER.
AB=Copper tube; *V*=Collecting vessel for condensed steam.
T=Water trap; *C*=Tube from boiler; *t*=Tube leading to condenser.

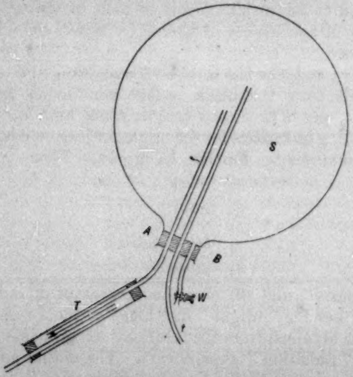

FIG. 2.—APPARATUS FOR MEASURING "TOTAL HEAT" FROM A SPHERE.
S=Sphere; *T*=Water trap;
t=Tube leading to condenser;
W=Tube for collecting condensed steam.

Steam was passed through the tube from a small boiler some distance away and separated from the tube itself by a wooden screen. A water trap *T* prevented water particles being carried over into the tube. The leading tubes from

A to *C* were thickly lagged with cotton wool. The water condensed from the steam was collected in a thin metallic vessel *V*, and after conditions had become steady was run off and weighed at intervals usually of 10 minutes. Any uncondensed steam was conveyed by a tube *t* to a condenser about a metre away. The temperature of the ascending air was read by a thermometer placed some 75 cm. immediately below the tube. Experiments were conducted with the tube (*a*) bright, (*b*) black, (*c*) bright, and the average of (*a*) and (*c*) compared with (*b*).

At the end of a series of experiments (20 or more) the tube was sawn off at *G* about 1 cm. from the collecting vessel *V* and the latter directly connected with the steam supply by inserting the cork *A* into the end *G*. A series of similar readings was then taken in order to obtain the correction for loss of heat from the collecting vessel. The remaining portion of the tube *A G* was that taken in the calculations as the emitting surface.

For the flat surface a flat copper vessel of dimensions approximately 70 × 15 × 1·4 cm. and total surface area 2443·6 sq. cm. was connected to the boiler, with the 15-cm. surface either horizontal or vertical, or at an angle inclined to the vertical. The results obtained showed too great a variation to form any definite conclusion. Alteration of slope often produced a great change in the total loss of heat. The irregularity of shape may have produced a considerable and irregular change in the flow of air currents. In all cases, however, the total loss of heat per square centimetre was greater than in the cylinder of largest radius of curvature considered.

The spherical surfaces employed (*see* Fig. 2) were thin glass flasks, which were coated externally with (*a*) aluminium varnish, (*b*) "dead black." Steam entered through a water trap *T*, the uncondensed portion leaving through tube *t* to a condenser, and the condensed steam being collected at intervals of 10 minutes from the tube *W*. Correction for loss of heat in this case is unnecessary, no "collecting vessel" being employed, and the water trap being close up to the spherical enclosure.

All parts below *AB* were lagged with cotton wool. Separate experiments in radiation were necessary to obtain the ratio for "aluminium" and "dead-black" surfaces. This was done by coating one half of the flask itself with aluminium and dead-blacking the other half, and employing thermopile and galvanometer in the usual way.

IV. EXPERIMENTAL RESULTS.

A. *Cylindrical Surfaces.*

(1) *Total losses of heat.*—Steam was passed for about an hour before any measurements were taken. The slope of the tube was about 1 in 7. The current of air upwards was at the rate of about 1 ft. per second as tested by tobacco smoke.

Table I. gives results for the four tubes and for the flat surface.

TABLE I.

	Tube 1.	Tube 2.	Tube 3.	Tube 4.	Flat Surface.
Length (cm.)	61·0	65·3	77·1	72·0	...
Diameter (cm.)	0·62	1·27	2·54	5·12	...
Surface area (sq. cm.)	118·9	262·2	616·4	1158·0	2444·0
Surface temperature (°C.)	97·7	98·3	99·3	99·65	99·7
Heat per sec. per sq. cm. per 1°C. excess	...	...	...	...	...
Calories × 10^{-6} black	495	362	280	222	317
Calories × 10^{-6} bright	432	285	208	155	215

In the last two lines of figures corrections have been made for the loss of heat from the collecting vessels (10 to 20 per cent.).

The degree of consistency of results is shown in Table II., obtained for the tube of diameter 1·27 cm.

TABLE II.

Tube bright.

Heat per sec. per sq. cm. per 1°C. excess. Calories × 10^{-7}.	Temperature excess. °C.
2869	75·6
3012	78·0
2928	79·7
2882	79·8
2819	76·7
2780	80·2
2743	78·0
2751	79·3
Mean Result .... 2848 × 10^{-7} calories.	

(2) *Relative losses by radiation for "bright" and "black" surfaces.*—One side of a "Leslie" cube was bright copper, another was coated with the same dead black as used in the

total heat experiments. Boiling water was poured into the cube and the radiation measured alternately from "black" and "bright" surfaces. The mean ratio was found to be 5·37 : 1, *i.e.*, "*b*"=5·37.

(3) *Collected results for the four tubes.*

TABLE III.

Diameter cms.	Total loss of heat "h" per sq. cm. per sec. per 1°C. (calories × 10^{-6}).		Loss due to convection (c) (calories × 10^{-6}).	Radiation loss (calories × 10^{-6}).	
	Black (h_1)	Bright (h_2)		Black (r_2)	Bright (r_2)
0·62*	495	432	416·0	79·0	16·0
1·275	362	285	267·5	94·5	17·5
2·545	280	208	182·0	98·0	16·0
5·12	222	154	139·5	82·5	15·5

* A different "black" for which "b" = 5·0 was used with this tube.

Equations (1) and (2) give the following percentages for the radiation losses.

TABLE IV.

Diameter of tube. cms.	Percentage of radiation. Black surface.	Percentage of radiation. Bright surface.
0·62	16·0	3·7
1·275	26·1	6·1
2·545	35·0	7·7
5·12	37·1	10·1
Flat surface	39·5	10·9

B. *Flat Surfaces.*

Table V. gives collected results for the flat surface with its length nearly horizontal and breadth vertical.

TABLE V.

Total area of flat surface	....	2443·6 sq. cms.
Total loss of heat (h)	black	317 × 10^{-6} calories.
	bright	215 × 10^{-6} ,,
Loss due to convection	....	192 × 10^{-6} ,,
Loss due to radiation	black	127 × 10^{-6} ,,
	bright	23 × 10^{-6} ,,
Percentage loss due to radiation	black	39·5 per cent.
	bright	10·9 per cent.

C. *Spherical Surfaces.*

(1) *Total losses of heat (radiation and convection).*

TABLE VI.

Diameter cms.	Effective area of surface. sq. cms.	Total loss of heat per sq. cm. (h) per 1°C. excess (calories × 10^{-6}).		Value of "a" $\left(=\frac{h_1}{h_2}\right)$
		Black.	Bright.	
7·55	182·0	405	311	1·30
10·07	314·3	371	288	1·29
14·0	611·7	343	257	1·33
16·5	849·2	333	247	1·34

(2) *Ratio of radiation losses (black: aluminium).*—A large spherical flask of diameter 14 cm. was coated on one side with aluminium paint, the other dead blacked. The value of "b" was found to be 2·37.

Taking the value of "a" from Table VI. we obtain the following values for convection and radiation for the spherical surfaces. (*See* Table VII.)

TABLE VII.

Radius cms.	Loss due to convection per sq. cm. per 1°C. excess. (calories × 10^{-6}).	Loss due to radiation per sq. cm. per 1°C. excess. (calories × 10^{-6}).		Percentage loss due to			
				Convection.		Radiation.	
		Al (r_1).	Black (r_2)	Al.	Black.	Al.	Black.
7·55	243	68	162	78·1	60·1	21·9	39·9
10·07	226	62	145	78·8	61·1	21·2	38·9
14·00	196	61	147	75·5	57·1	24·1	42·9
16·50	186	61	147	75·2	56·1	24·8	43·9

The percentage of radiation is seen to increase only slightly with increase of radius of curvature within the limits observed.

V. DISCUSSION OF RESULTS.

(*a*) *Comparison of "heat lost by convection" with the diameter of cylinder.*—It has been shown in Russell's Paper* that the convection of heat from a cylinder per square centimetre of surface must be inversely proportional to the square root of the diameter. The proof applies strictly only to the case of a cylinder immersed in a cooling fluid which moves with constant appreciable but not excessive velocity in a direction at right angles to the length of the cylinder. The fluid is supposed to be opaque to heat rays, non-compressible, of very low thermal conductivity and to possess no viscosity. The cases tested in the present Paper can only be expected, therefore, to give approximate agreement with this theory.

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* Loc. cit.

Table VIII. shows the relation actually found between the heat lost (by convection only) and the diameters of the cylinders. The product of the "convection" and the square root of the diameter appears to be approximately constant, *i.e.*, $c \propto \frac{1}{\sqrt{d}}$ within limits of experimental error.

TABLE VIII.

Diameter d. cms.	$\sqrt{d}$.	Heat lost by convection. (c) (calories × 10^{-6}).	$c\sqrt{d}$.
0·62	0·787	416·0	327
1·275	1·129	267·5	302
2·545	1·595	182·0	290
5·12	2·263	139·5	316

(*b*) *Convection from spherical surfaces.*—In the case of a *sphere* the figures indicate that the convection per square centimetre is inversely proportional to the *cube root* of the diameter. (*See* Table IX.)

TABLE IX.

Diameter d. cms.	Convection loss per sq. cm. (c). (calories × 10^{-6}).	$C \times \sqrt[3]{d}$.
7·55	243	456
10·07	226	488
14·0	196	472
16·5	186	472

That is $c \propto \frac{1}{\sqrt[3]{d}}$.

(*c*) *Losses by radiation.*—The percentage radiation losses as given by the experiments described above are considerably higher than those obtained in a previous paper for wires of 1 mm. diameter. This is partly accounted for by the greater difference of temperature between the hot surface and the ascending gas (80°C. compared with 10°C. or 12°C.), but chiefly because the total losses are smaller per square centimetre, while the radiation remains approximately constant. Radiation losses as a percentage of "total heat" emitted are given in Table X.

TABLE X.

Surface and radius of curvature.	Percentage of radiation loss.	
	Bright.	Black.
Cylinders 0·32 cm.	3·7	16·0
0·64	6·1	26·1
1·28	7·7	35·0
2·56	10·1	37·1
Flat surface	10·9	39·5
Spheres 3·77 cm.	21·9	39·9
5·03	21·2	38·9
7·00	24·1	42·9
8·25	24·8	43·9

The radiation is given—approximately at any rate—by Stefan's fourth power law, $R=k(\theta^4-\theta_0^4)$. It is interesting to compare the radiation losses as given by this equation for the two cases,

(*a*) where $\theta=100°\text{C.}$; $\theta_0=10°\text{C.}$,

and (*b*) where $\theta=20°\text{C.}$; $\theta_0=10°\text{C.}$

The ratio is $$\frac{\overline{373}^4-\overline{283}^4}{\underline{293}^4-\underline{283}^4}=13{\cdot}5.$$

Thus the increase in radiation is only 13½ times more in the first case than in the second.

At the same time the convection (by Newton's law) would increase 9 times.

In cases where the temperature excess is not very high and the percentage of radiation is small (as in wires and narrow tubes), it follows that little error is made in assuming the linear law for *the total loss of heat* from the surface. Results given in Tables III. and VII. indicate that the loss of heat per square centimetre by radiation alone is independent of the curvature of the surface. That is to say that the increase of loss of heat due to decrease of radius of curvature is entirely due to increase in convection.

In the case of the flat surface the radiation per square centimetre appears to be greater than in the curved surfaces, but owing to the variation in results obtained for slight changes of slope, &c., no great weight can be attached to this.

The experiments were performed at the Wandsworth Technical Institute, most of them about four years ago, but owing to military duties publication has been delayed until now.

SUMMARY.

An experimental determination has been made of the amounts of "natural" convection and radiation from cylindrical and spherical surfaces heated in air at atmospheric temperature and pressure to a temperature of about 100°C.

It is shown that the convection alone is inversely proportional to the square root of the diameter of a cylinder, and to the cube root of the diameter of a sphere.

The percentage of radiation to total heat emitted is given in the following table :—

Surface.	Diameter cms.	Percentage of radiation.	
		Bright.	Black.
Cylinders	0·62	3·7	16·0
	1·275	6·1	26·1
	2·545	7·7	35·0
	5·12	10·1	37·1
Flat surface	...	10·9	39·5
Spheres	7·55	21·9	39·9
	10·07	21·2	38·9
	14·00	24·1	42·9
	16·50	24·8	43·9

ABSTRACT.

The relative and absolute amounts of radiation and convection from surfaces heated to about 100°C. in air were measured as follows: steam was passed through the cylinders or spheres, the surface temperature being measured by a thermo-junction. The total amount of heat lost from the surface was determined from the equivalent mass of steam condensed. This was done with the surface (*a*) "bright," (*b*) "dead-black." The relative amounts of radiation alone from these two surfaces were then found with the aid of a thermopile. If h_1 is the total heat lost, and r_1 the radiation, per square centimetre per second per 1°C. excess of temperature for the "black" surface ; h_2 and r_2 corresponding quantities for the "bright" surface ; and c the convection in each case, then if $h_1=ah_2$, $r=br_2$,

$$\frac{r_2}{h_2}=\frac{a-1}{b-1},\quad \frac{r_1}{h_1}=\frac{b(a-1)}{a(b-1)}.$$

The numerical values of the radiation, convection and total heat are then easily calculable, and are given in tables. It is found that the amount of *convection* per square centimetre is inversely proportional to the square root of the diameter of a cylinder, and to the cube root of the diameter of a sphere.

DISCUSSION.

Mr. C. R. **Darling** pointed out that the numerical results contained some considerable discrepancies. For instance, in Table 2, for an excess of 75·6 deg. the heat loss was 2,869 units, while with a higher temperature excess of 80·2 deg. it was only 2,780. Something must be wrong, and he distrusted the use of steam condensation methods of measuring the heat lost. Water traps were never satisfactory. Why not use electrical heating? A second point was the method of measuring the surface temperature by inserting a thermo-junction in a little notch. Some experiments he had once made showed that the conditions just under the surface differed considerably from those at the actual surface when heat is escaping. Had the author tested the results mentioned by Langmuir ("Journ." Am. Electro-Chem. Soc., Vol. 23, 1913)? Langmuir states that the convection loss is equal to the heat conducted through a layer of air of particular thickness surrounding the hot body. Further, the loss from a plane disc is 10 per cent. more when the disc is horizontal than when it is vertical. In the horizontal case the loss from the lower surface is 50 per cent. less than from the upper. Langmuir found that the phenomena of natural convection did not agree with Russell's formulæ, but forced convection did.

Dr. **Ezer Griffiths** thought the Paper of interest insomuch that it contributed some data for cylinders of moderate diameters and for spheres. The experimental work of Langmuir referred to by the last speaker did not cover the same ground, since it dealt only with fine wires and one flat surface. He agreed that the method of steam heating was not altogether satisfactory, although most of the data relating to steam pipes had been obtained by means of it. When electrical methods were employed the difficulty was the large amount of energy which had to be supplied. For example, in some experiments on convection loss now in progress it was necessary to supply about 18 kw. of electrical energy to heat up a cylinder 9 in. in diameter by about 4 ft. long to a temperature of about 450°C. He did not think that the experiments of Dr. Barratt proved conclusively the relation between diameter and convection loss per unit area given in the Paper. It was a well-known fact that the change of heat loss with diameter was greatest with cylinders of small diameter, such as wires, but above about 3 in. in diameter the change was small and the form of the curve connecting heat loss (convection per unit area), and diameter for the complete range of diameters (wires of 2 mils to cylinders 9 in. diameter) was of the form which he illustrated on the board.

Capt. **Dunsheath** said that at present millions of pounds were invested in cables for conveying electric power, and more information on the heat loss from such cables would be of the greatest practical value in determining the necessary thickness of copper to employ. The present experiments, however, did not carry us far in this respect, as the cable is rarely suspended in free air. It would be useful if the experiments could be carried out with the cylinder in contact with a plane or lying in a duct. As regards the measurement of surface temperature, he thought the conduction along the wires of the couple would affect this unless they were arranged to be in thermal contact with the surface for some distance from the junction. How was the effects of draughts eliminated? Were any steps taken to measure the air currents existing before the cylinders were heated?

Mr. G. D. **West** mentioned the importance of the inclination of the surfaces to the vertical. With wires in tubes a great deal depended on whether the tubes are horizontal or vertical. If the author could give us data on this point, we should be a step nearer measuring the conductivity of the surrounding gas, which, at low pressures, is more important than convection.

Mr. F. E. **Smith** agreed with what had been said regarding the inefficiency of steam traps. It appeared to him that measurements of this type could be carried out by using very thin walled tubes conveying an electric current.

By measuring the resistance of the tube, both the temperature and the energy consumed could be determined.

Dr. **Russell** (communicated): As a knowledge of the relative amounts of heat convected and radiated from hot bodies is of great practical value to the engineer, the authors are to be congratulated on having made a notable contribution to our knowledge of the subject. It is satisfactory to find that the heat convected from the cylinders used by the authors varies approximately inversely as the square root of their diameters, although only free convection is considered. I think that we may infer that the mean velocities of the convection currents of air round all the cylinders is approximately the same. I was surprised at the large amount of heat convected per unit area from the spherical glass vessels as compared with that convected from cylinders of equal diameter. For the same difference of temperature between the bodies and the air I should have expected, from theoretical considerations, that the heat convected per square centimetre from a cylinder would be $2\sqrt{2}/\pi$—that is, 0·9 of that convected from a sphere of equal diameter. It appears from the authors' results that the heat convected from a cylinder of 7·55 cm. diameter is only about 0·45 of that convected from a sphere of the same diameter—that is, it is only about half the theoretical value. On certain theoretical assumptions, also, the heat carried away per square centimetre from a sphere by forced convection varies inversely as the square root, and not inversely as the cube root of the diameter. Further research on the cooling of spheres, more especially with forced draught, would be very interesting. I should expect that the heat convected per unit area would vary as the square root of the velocity of the draught, and inversely as the square root of the diameter. The heat also convected per unit area from a sphere should be about 10 per cent. greater than that convected per unit area from a cylinder of the same diameter placed with its axis perpendicular to the same draught.

Dr. **Barratt** communicated the following reply to the points raised by speakers: The "heat loss" mentioned by Mr. C. R. Darling is in each case the result *per degree excess.* My reason for employing a steam condensation method was for the purpose of simplicity in apparatus. As regards "surface temperature" (criticised also by Capt. Dunsheath), each temperature measured was in the region of 100°C., and was directly compared with the temperature of steam at atmospheric pressure. I am convinced the temperatures measured were pretty accurate. I read Langmuir's results some years ago. Natural convection depends so much on the size and disposition of apparatus, and the nature of the surroundings, that the problem of a flat surface appears to be almost insoluble experimentally. In reply to Dr. Ezer Griffiths' remarks as to the change of heat loss with curvature in cylinders of large diameter, it is worthy of notice that in this case the radiation loss becomes relatively very large. When the convection loss alone is considered (this being thoreetically nil for an infinite flat surface), the inverse square root law appears to be approximately applicable. No special precautions were taken to exclude draughts beyond those mentioned in the Paper, and these probably accounted to a great extent for the discrepancies shown in the results. Mr. F. E. Smith's suggestion is a good one if one could employ tubes of a pure metal (*e.g.*, platinum), where the temperature resistance curve remains constant. The question raised by Mr. G. D. West of convection losses from wires in tubes is a difficult one to test experimentally, owing to the impossibility of obtaining constancy in the natural convection stream.

THE FIFTH GUTHRIE LECTURE.

The Anomaly of the Nickel-Steels.

Delivered by DR. CHARLES EDOUARD GUILLAUME, *Director of the Bureau International des Poids et Mesures, Sèvres.**

APRIL 23, 1920.

Discovery of the Anomaly.

IT is close on 30 years since John Hopkinson—whose tragic death still remains to us a sad memory—discovered a remarkable fact: a specimen of an alloy of iron and nickel containing 24 per cent. of nickel was soft and non-magnetic; but, on plunging it into solid carbon dioxide, it became hard and magnetic, its volume undergoing at the same time an increase of about 2 per cent.

This steel might be said to be of a decidedly anti-metrological character; and, so far as the purpose of my researches at the Bureau International des Poids et Mesures was concerned, I naturally sought to give a wide berth to all phenomena of the kind.

My investigations had for their object the perfecting of standards of length. The General Conference of 1889 had distributed a number of admirable standards amongst those States represented in the Metric Convention. These standards were constructed of the platinum-iridium alloy discovered by Henri Sainte Claire Deville, the preparation of which was entrusted to Johnson Matthey. But the great cost of these standards debarred their use for most requirements; whilst, on the other hand, the usual standards were far from reliable.

I therefore undertook the search for a metal or alloy which, whilst little inferior in general metrological qualities to platinum-iridium, should be considerably lower in price.

A preliminary investigation resulted in establishing the remarkable metrological properties of pure nickel; and, even to this day, it is this metal to which we have recourse when in need of a standard which shall be unoxidisable, invariable with time, rigid, and of moderate expansibility. A difficulty, however, deterred me from using it in all cases. The chief desideratum was to produce a geodesic standard of a length of four metres; but no firm of nickel manufacturers would undertake to supply a bar of that length perfectly sound and free from cracks.

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* The Lecture was delivered in English. The translation of the French MS., made by Dr. D. Owen, has had the advantage of the Author's perusal.

In 1895, M. J. R. Benoit, at the request of the Technical Section of Artillery at Paris, was engaged in the examination of a standard prepared from an iron alloy containing 22 per cent. of nickel and 3 per cent. of chromium. He found it to possess a coefficient of expansion approximately equal to that of brass. This alloy was non-magnetic, and appeared almost certainly to belong to the same class as Hopkinson's steel before its transformation by cooling. But the anomaly discovered by M. Benoit, although of great interest from the physical point of view, offered no particular prospect of advantage to metrology.

The question took a new aspect when, in the following year, I came across a new and quite unexpected fact, closely connected with those to which I have alluded. A bar of steel containing 30 per cent. of nickel, received at the Bureau International, was found to have a coefficient of expansion about two-thirds of that of platinum, and at the same time to be magnetic. Thus opened a promising line of investigation, which was eagerly pursued.

The question of expansibility is fundamental in metrology. Before the discovery of the anomaly just stated, every physicist would have affirmed that it was hopeless to attempt to solve the problem of finding alloys of a much lower coefficient of expansion than the values then known, since the rule of mixtures was regarded as holding practically exactly in all cases.

My first care was to verify the trend of the variation of expansibility of these alloys as a function of chemical composition. This precaution was not without warrant, in view of the possibility of a discontinuity occurring in the region between the non-magnetic alloy containing 22 per cent. of nickel, and the magnetic alloy containing 30 per cent. Experiments made on two alloys, on opposite sides of the last mentioned as regards nickel content, and which were supplied by the Company of Commentry-Fourchambault et Decazeville, served to establish the fact of continuity.

Classification by means of Magnetic Properties.

The precise measurement of the coefficient of expansion is a long and delicate one. Since one was in presence of an anomaly affecting, doubtless, all the properties of the new alloys, it became necessary to seek for methods of greater convenience. The examination of the magnetic susceptibility naturally suggested itself; where it is a question merely of ascertaining the presence or absence of ferro-magnetism, this test is of the simplest. The outcome of the study of this effect was the recognition of the existence of two distinct transformations : the one irreversible, being that which Hopkinson discovered ; the other reversible, and constituting a new phenomenon.

Other observers, notably MM. Osmond, Louis Dumas, Pierre Weiss and his pupils, Nagaoka, and Honda have worked out the details.

A simple diagram enables us to see at a glance the complete course of transformations of the series of ferro-nickels as related to their magnetic properties.

Commencing with pure iron, the variations resolve themselves along two branches which gradually get further apart (Fig. 1). At the intersection, from top to bottom, of the lower curve, magnetism appears, and increases up to a certain limit. On reheating the alloy, it begins to decrease at a given temperature, and definitely disappears along the upper curve. On the contrary, the passage along the single branch of the curve denotes, in the case of alloys of higher nickel content, that the appearance of ferro-magnetic properties on cooling, and their disappearance with rise of temperature, occur at one and the same temperature.

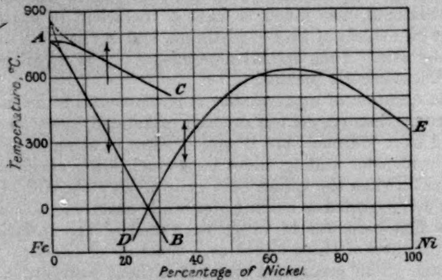

FIG. 1.

Temperature of magnetic transformation of the nickel-steels as a function of composition. The two branches *AC* and *AB* refer to the irreversible state ; the single branch *DE* to the reversible state.

The point of crossing of the curves of the two categories has a special significance ; additions of carbon, chromium, manganese* lower markedly the temperature of an irreversible transformation, but have much less influence on reversible transformations. It is thus possible to follow a reversible transformation in a normally irreversible region. At a time when the only copious source of liquid air available was at the Royal Institution, I succeeded, thanks to the courtesy of Sir James Dewar, in demonstrating the existence of reversible magnetic properties in the alloys containing 22 per cent. Ni and 3 per cent. Cr.

On the other hand, on the right of the crossing a moderate cooling still leaves the magnetic condition completely reversible; whilst a more considerable cooling brings about the transformation and renders it irreversible.

Let us now suppose a third axis perpendicular to the other two, along which is plotted the value of the susceptibility;

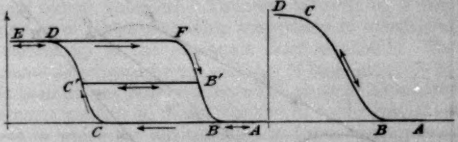

FIG. 2.

Variation of magnetic susceptibility of nickel-steels with temperature, in the irreversible and reversible regions respectively. (Ordinates denote susceptibility, abscissæ temperature.)

and, in the solid diagram thus obtained let a vertical section be taken. The variation of susceptibility with temperature will have, in the irreversible region, the slope given by the curves *ABCDEF*, and in the reversible region that represented by the unique curve *ABCD* (Fig. 2).

One can, in the case of the irreversible condition, arrest the cooling at any moment, then reheat the alloys, thus conserving the magnetic properties constant, as shown by the line *C'B'* (Fig. 2).

All the properties of the alloys with which we are concerned are bound up with these transformations.

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* Manganese, alone, lowers the temperature of transformation of the alloys which it forms with iron; on this property depends the practical realisation, by Sir Robert Hadfield, of the manganese steels, with their very remarkable properties.

The clue being thus found, I gave my utmost efforts to the study of change of volume, which constitutes the point of capital metrological interest of these alloys.

Changes of Volume.

Irreversible Changes.—Knowledge of the irreversible changes which these alloys undergo was necessary, particularly with a view to fixing the limiting value. I have studied their different aspects in the binary alloys of nickel-iron, and their ternary alloys with chromium, copper, &c. It has only been possible for me to trace the lower part of the cycle; complete cycles were realised much later by M. P. Chevenard.

When a rod of an irreversible alloy is cooled from a high temperature, it contracts according to a straight line law, as seen in *ABC* (Fig. 3); at a certain temperature the contraction becomes less rapid, and finally a progressive increase of volume sets in, which goes on steadily until the transformation is complete. After that, the contraction resumes its regular reversible path *DE*.

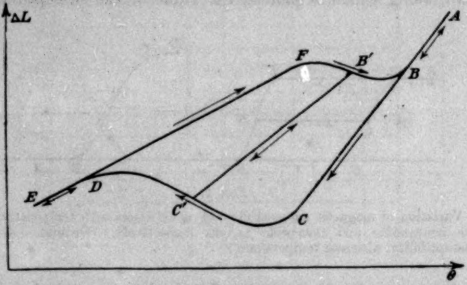

FIG. 3.
Expansion curve of a nickel-steel in the irreversible state.

If the rod is again heated, it is observed to expand along *EDF*, and then contraction sets in, which restores it to the line *AC*. If the cooling is arrested at the point *C′*, and the bar reheated, the changes that occur at first follow the line *C′B′*, and then continue along the curve *B′BA*, just as in the preceding case. The inclination of the line *AC* is about 18×10^{-6}, and that of *EF* between 10 and 11×10^{-6}. The former figure represents the expansibility of a non-magnetic alloy of iron-nickel-chromium; the latter the value in the case of the ordinary steels. Between these values one can obtain any intermediate rate of expansion by simply arresting the transformation at a suitable point on one or other of the branches *CD* or *FB*.

Along the curve *AC* the steel is in a state which is stable from the high temperature downwards (stable à chaud); along *EDF* it is in a state stable from low temperature upwards (stable à froid).

Reversible Changes.—Experiments on the reversible alloys have been much more extensive.

A preliminary investigation was made to establish the curve of the anomaly as a function of the nickel content, without any regard to the presence in slightly variable quantity of manganese, carbon and silicon—which left the curves to a certain extent indefinite. This was followed by a study of the series of alloys containing proportions of manganese and of carbon up to the maximum limit attainable. The coefficients as affected by the addition of these elements having thus been determined over the whole range of nickel content, it became possible to reduce the results to the special case of alloys in which the additions are in the uniform percentage proportions 0·4 Mn, 0·1 C, and which I have called "typical alloys."

These alloys, rolled when hot and allowed to cool in air, may be said to be " in the natural state."

The expansion of a rod being represented by the equation $l_\theta = l_0\,(1+\alpha\theta+\beta\theta^2)$, we shall call the quantity $\alpha_\theta = \alpha + 2\beta\theta$ the *true* coefficient of expansion at $\theta°$; it is also the *mean* coefficient of expansion between 0° and $2\theta°$. We shall term β the quadratic or curvature coefficient.

The expansion of most of the metals is well represented, over a large range of temperature, by an equation of the form given above; in the case of the alloys with which we are most concerned, the same law applies sufficiently well over the narrow range, 0°-38°, within which I have worked with the comparator; but when the range is extended it soon becomes evident that an equation of the second degree insufficiently represents the facts. It is, however, convenient to preserve the same form of equation, attributing to β a value varying with temperature, and equal to half the curvature at any point of the curve of expansion.

The values of α_{20} and β_{20} for the typical reversible nickel-steels in the natural state are represented by the curves of Figs. 4 and 5. The straight lines *AB* connect the values of

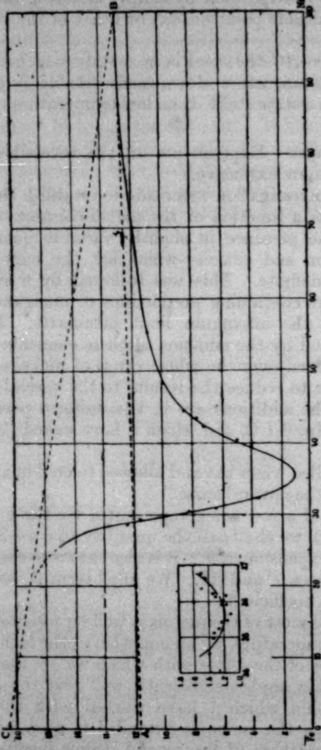

FIG. 4.

True expansibilities at 20°C. (in millionths) of the typical alloys of iron and nickel (containing 0·4 Mn and 0·1 C per cent.). Abscissæ denote the percentage of Ni. The inset represents the region of the minimum to a larger scale.

these coefficients for pure iron and nickel in the condition which is stable at low temperature, and thus make evident the magnitude of the anomaly of expansion. In regard to β, this anomaly is positive at first, then becomes negative. The same appears at first sight to be the case with *a*; but on tracing the line *CB*, which starts at the value which this coefficient possesses in the case of gamma iron, the anomaly is seen to be entirely negative. It will be shown later that the anomaly receives its full explanation in terms of this last fact. It may be observed, further, that the expansibilities of the irreversible alloys are limited by the straight lines *AB* and *CD*.

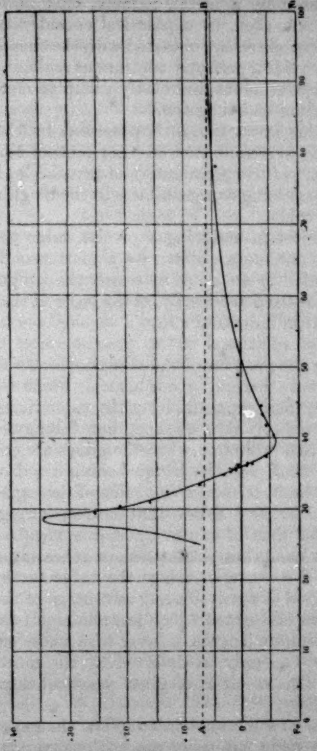

FIG. 5.

Values of the coefficient $10^9\ \beta$ of the typical alloys of iron and nickel (containing 0·4 Mn and 0·1C per cent.).

The anomaly is of considerable range, since the expansibility of the alloys varies in the extreme proportion of 1 to 15, the value at the lower limit being only a quarter of the least coefficient possessed by a metal, namely, that of tungsten. It may also be remarked, as a practical consideration, that these feeble expansibilities are obtained with the cheap metals: and that these provide a perfectly continuous scale, in contrast to that of the discontinuous and costly group of metals, such as iridium, tantalum and tungsten.

The lowest value ($a=1{\cdot}19\times10^{-6}$) possessed by the typical alloys corresponds to a content of 35·6 per cent. of Ni.

The generic name *Invar* (diminutive of invariable) has been given to alloys possessing expansibilities in the neighbourhood of the minimum value.

There is no object in assigning a precise value to the coordinates of the minimum point. As a matter of fact, the values quoted refer to an alloy in which the proportion of other elements was fixed arbitrarily, of the order of the average quantities present in industrial alloys; we shall see later that the proportions of additional metals present are of considerable influence on the position of the minimum. On the other hand, all treatment, thermal or mechanical, which the alloys undergo, modifies their expansibility; the value rising in case of heating followed by slow cooling, but falling when the cooling is rapid, and still more when the alloys are cold-rolled or drawn. It is thus possible, by quenching a rod and then drawing it to the limit, to reduce the value of the expansibility by an amount $1{\cdot}5\times10^{-6}$ below that corresponding to the natural state, and thus to confer upon it a negative value. Then, when this condition is attained, a reheating of some hours at 100°C., for example, brings the value to the immediate neighbourhood of zero. Taking advantage of these facts a method has been elaborated which has rendered possible the production of kilometre-lengths of invar wire whose expansion with temperature can only be detected by the most precise measurements. This result is of great practical importance, especially to geodesy.

It was a matter of some interest to follow the expansion of the alloys over a more extensive range of temperature than that permitted by the comparator, even at the sacrifice of some of the accuracy. In 1896, I was able to extend the measurements as far as 220°C., and to determine the manner in which β varies. The appearance of the curves immediately suggested a generalisation, which allowed of the prediction of phenomena before suitable methods for their study had been elaborated.

Even within the range of measurements by the comparator, it is evident from an examination of Figs. 1 and 5 that the high values of β correspond to the region of temperature within which the alloys pass gradually, on cooling, into the magnetic state. I thought that the same might hold in the case of those alloys whose transformation commences at a temperature above the limit which my experiments had attained; and I anticipated that, at an equivalent distance from the commencement of the appearance of magnetic properties, different alloys would be in approximately analogous states.

Taking, then, the values of the expansibilities over the particular range of my experiments, I plotted them against nickel content, and found that the resulting curve is of similar shape to that representing the expansibility of one and the same alloy through all the transformations which it undergoes over a large interval of temperature. Thus was obtained the "rule of corresponding states" for the nickel-steels. Employed with care—for it is not a precise law—it has proved of the greatest utility.

In order to apply this rule, we may start with the value of β appropriate to the alloys containing 70 per cent. Ni, at which percentage the temperature of magnetic transformation attains its maximum value; and pass progressively through the alloys of gradually diminishing nickel content, down to that containing from 26 per cent. to 27 per cent. Ni, for which the magnetic transformation occurs at the ordinary temperature (*see* Fig. 1). On plotting the values of β at ordinary temperature against nickel content a curve is obtained, which proves to be none other than that found on plotting the value of β for a single alloy against temperature over a considerable range.

The value of β, to begin with, is positive; and the corresponding portion of the expansion curve (*AB* of Fig. 6) is slightly concave upwards. We then enter the region of negative values of β, and the curve bends downwards (*BC*) thus passing into the region of low coefficient of expansion (*CD*). β then recrosses the zero line, and becomes rapidly positive; its value increases, reaches a maximum, then falls to the region of normal values, the coefficient of expansion remaining, however, very high (*EF*).

The series of values thus deduced directly from β is found to agree closely with that obtained on examination of the corresponding values of α. Thus, from a comparison of the two diagrams, the unity derived from the consideration of corresponding states is seen to be perfect.

The progress of technical metrology has, within the last few years, made it possible to examine the expansion of a great number of nickel-steels over a range of a thousand degrees. M. Chevenard especially has made an extensive series of investigations, which have resulted in the verification of the general forecasts. They have at the same time shown that the expansion curve of the alloys gradually changes character :

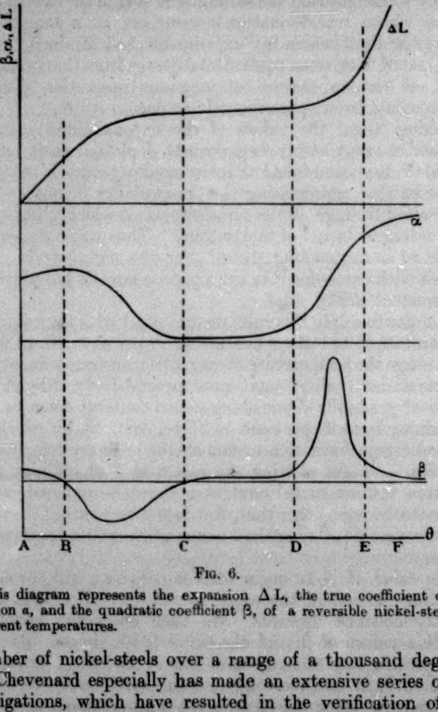

Fig. 6.

This diagram represents the expansion Δ L, the true coefficient of expansion α, and the quadratic coefficient β, of a reversible nickel-steel, at different temperatures.

with increasing nickel content the range of temperature over which the volume remains practically constant becomes smaller and smaller ; the expansion curve concurrently altering in slope so that in the alloys of large nickel content no very low value of the coefficient is to be found anywhere within the total temperature range.

Effect of Additions.—A glance at the curve of Fig. 4 shows that in the anomaly under consideration the rule of mixtures, flagrantly at fault in regard to the two constituent elements iron and nickel, is no better respected when we pass to the mixture of invar with one or other of its components. Except in the immediate neighbourhood of the minimum, the coefficient of expansion rises much more rapidly, whether on addition of nickel or of iron, than according to the rule of mixtures.

It would, therefore, appear probable that the same would apply when additions other than those of an excess of its principal constituents are made to invar.

In this connection the study of changes of expansibility on addition of other components is of considerable practical interest, since it may be desired to modify the general properties of alloys possessing the anomaly, in order to adapt them to particular classes of application. In preparing castings, for example, one would increase the manganese content ; or, again, if it were desirable to raise the elastic limit, one would add chromium or carbon.

I have carried the study of these alloys up to the utmost practicable limit of addition. Anticipations have been confirmed, in the sense that the minimum expansibility is always found to be raised ; but, at the same time, it has been found that the minimum abscissa travels along the axis, either towards the iron or towards the nickel.

The two diagrams of Figs. 7 and 8 permit of following the displacement of the minimum, both as regards abscissæ and ordinates. The rest of the curve rises as a whole, so as gradually to obliterate the depression representing the anomaly. But, on account of the displacement in abscissæ, in the three cases in which this is very rapid (Mn, C, and Cu), there occurs, to the right of the minimum for Mn, to the left for C or Cu, a narrow range of contents within which the ternary alloys may have a coefficient of expansion slightly less than that of the binary alloys.

It is just as well to remark that the diagrams are drawn in one respect arbitrarily. They are given in terms of nickel content as abscissæ; if, however, they are referred to iron, the abscissæ would be modified by an amount equal to that of the addition considered. Take, for instance, the binary alloy containing 36 per cent. of Ni and 64 per cent. of Fe; with 8

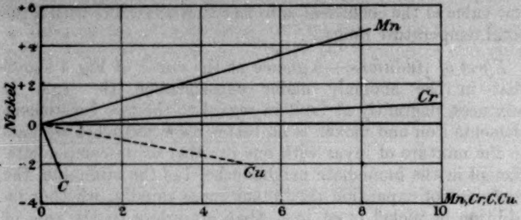

FIG. 7.

Displacement of the abscissa of the minimum of expansibility of the nickel-steels as a function of the proportions of a third constituent. [Abscissæ denote percentage of the additional constituent (Mn, Cr, Cu, C); ordinates denote the changes in percentage of nickel content, starting from the content corresponding to the minimum in the typical alloys.]

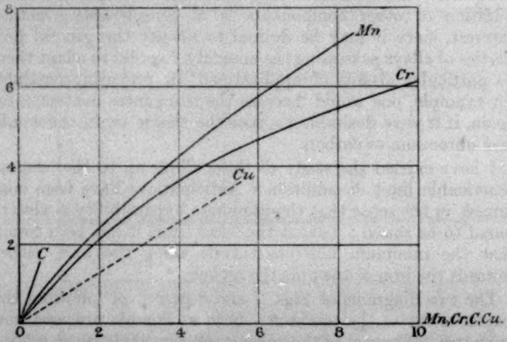

FIG. 8.

Change in the minimum expansibility of the nickel-steels when a third constituent is added. (Abscissæ denote percentages of the added element; ordinates denote the changes in $10^6\alpha_{20}$, starting from the minimum value appropriate to the typical alloys.)

per cent. of Mn the alloy having the same abscissæ in nickel-content would have, referred to iron, an abscissa of 56.

A diagram drawn according to the method advocated by Guthrie avoids this indefiniteness. The basis of this diagram, as is well known, lies in the property of the equilateral triangle, according to which the sum of the distances of any point in the interior from the three sides is constant, being equal to the height of the triangle. The distance from each of the three sides represents the percentage of the corresponding metal, whilst the ordinate erected perpendicular to the plane of the triangle represents the numerical value of the property under consideration. (At this point the Lecturer showed the Guthrie diagram relating to the ternary alloys of Fe, Ni and Mn, and expressed his pleasure in paying homage to the memory of the founder of the Physical Society, under whose auspices the lecture is delivered).

Elastic Properties.

The Elastic Modulus.—My researches have related mainly to Young's modulus. I have ascertained that the transformation of the irreversible alloys by cooling, whilst raising the elastic limit to a considerable extent, reduces the modulus by a tenth of its value. In the case of the reversible alloys, the modulus falls concurrently with the coefficient of expansion, and attains a value of about a quarter less than that which would follow from the rule of mixtures. The minimum occurs in the invar region, where the modulus has a value of about $1{\cdot}4 \times 10^{12}$C.G.S.

Temperature-variation of the Modulus.—The anomaly in regard to the elastic properties is very remarkable, whether expressed in terms of Young's modulus or of the modulus of torsion.

The examination of the variation of Young's modulus with temperature was carried out on watches provided with springs of the alloy selected; most of the observations were made, under my direction, by Paul Perret, or by the Société des Fabriques de Spiraux Réunies. From the results obtained over a range of 30 degrees C., interpreted in the light of the rule of corresponding states, I have been able to draw representative diagrams, just as in the case of expansibility. These deductions were verified later by Félix Robin.

In regard to torsion, a rapid examination sufficed to establish the existence of effects of the same order as in the case of flexure. The investigation was extended by M. Ch. Eug. Guye to the case of invar; and M. Chevenard for the first time made full detailed measurements.

The period of oscillation of a balance-wheel controlled by a spring depends mainly, so far as temperature is concerned, on the variation of Young's modulus; the expansion of the spring only would cause the watch to gain with rise of temperature, whilst expansion of the balance-wheel only would cause it to lose to about the same extent. In the case, for example, of a steel spring associated with a balance-wheel of brass, the effects of expansion would practically compensate each other, and the going of the watch would have a temperature-coefficient one-half of the thermo-elastic coefficient of the spring.

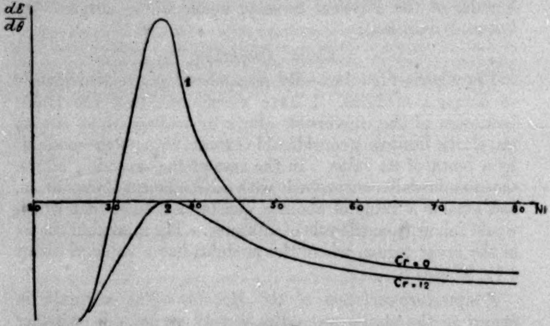

FIG. 9.

Temperate-coefficient of elastic modulus of the nickel-steels as a function of nickel constant. (Curve 1 refers to the pure nickel-iron alloys; curve 2 to alloys containing added elements equivalent to 12 per cent. of chromium.)

It will always be possible, on making a small correction for the outstanding effect of expansion, to deduce from the rates of the watch the value of the thermo-elastic co-efficient.

Curve 1 of Fig. 9 represents the thermo-elastic coefficients (temperature-coefficients of Young's modulus) in the neighbourhood of 20°C.; and curve 1 of Fig. 10 the values of the modulus for a particular alloy over a large range of temperature, as deduced from the preceding diagram by application of the rule of corresponding states.

The first diagram is striking on account of its similarity to that of the expansibility. It is evident that if one were to replace the coefficient of variation of Young's modulus by that of expansion, and shear one of the curves parallel to the axis of ordinates, the curves would practically coincide. The correspondence is so exact that it has proved possible, as we shall presently see, to deduce the existence of certain peculiarities in the thermo-elastic properties of the nickel-steels when given their coefficients of expansion.

The main fact disclosed by the curves of the thermo-elastic coefficient is as follows: Between two limits of content at a given temperature (or between two limits of temperature at the same content), the thermo-elastic coefficient possesses a positive value; so that between these limits a strip of the alloy, first put under flexure and then raised in temperature, would tend to straighten itself. This fact was directly verified by Marc Thury in 1897, in the case of a strip of invar. The value of the thermo-elastic coefficient in this case was $+0{\cdot}46\times10^{-3}$, that is to say, of the same order of magnitude as that of bronze, but with sign reversed.

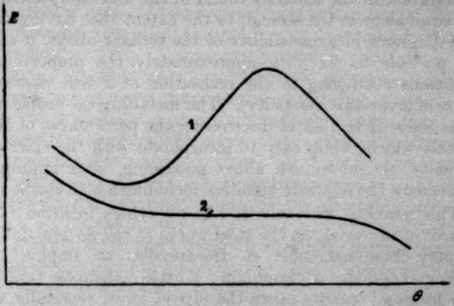

FIG. 10.

The elastic modulus (E) of a nickel-steel as a function of temperature (θ). (Curve 1 relates to the pure iron-nickel alloy; curve 2 shows the effect of the addition of 12 per cent. of chromium.)

Just as in the case of the expansibility, the initial part of curve 1 (Fig. 10) corresponds, at ordinary temperature, to alloys of high nickel content. The minimum is associated with the second intersection with the zero axis (Fig. 9), the rise with the invar region, and the maximum with the first intersection.

The existence of both the maximum and the minimum of the modulus have given rise to numerous applications wherever there was need of a feebly variable modulus. But in view of the requirements of technical metrology it was deemed desirable to search for a more satisfactory solution.

In order to do this without undue work of a preliminary nature, I made use, as I have said, of the similarity of the curves of expansibility and thermo-elastic coefficient. The diminution in the anomaly found in the first case should find its equivalent in the second, to the extent that having drawn the diagrams of expansibility of the ternary alloys, it should be possible to fix very approximately the proportions of additions conducing to the realisation of a true *elinvar* (or alloy of invariable elasticity). The metallurgical problem was thus reduced to that of discovering the percentages of added metals which, while easy to incorporate with the chief components, should furnish alloys possessing, after appropriate treatment, the requisite qualities, including a high elastic limit.

This problem was solved, like all others incurred in the course of my work, at the steel works of the Société de Commentry-Fourchambault et Decazeville, at Imphy. The principal addition is chromium; a little manganese facilitates the forging; carbon raises the elastic limit, especially when accompanied by a hardening metal. A total addition of about 12 per cent. of foreign metals lowers the curve to that of No. 2, Fig. 9, making it tangential to the zero axis. At the same time the rule of corresponding states indicates that the successive values of the modulus for the same alloy will be represented by curve 2 of Fig. 10. The verification of this inference, made over a range of some 30 degrees, has proved quite exact. In the case of elasticity of torsion M. Chevenard has shown that the actual form of the curve agrees with that predicted over a range of nearly 300 degrees.

Progressive and Transitory Variations.

The properties of the nickel steels, so important in metrology, of which a brief exposition has just been given, are, unfortunately, accompanied by a slight degree of instability, which necessitates the use of certain precautions in the application of these alloys. The main effects referred to will be exemplified in the case of invar.

A bar of this alloy which has been forged and allowed to cool in air, then maintained at, say, 100°C., experiences at the outset a rapid increase of length, which can be followed for about 100 hours. If now the temperature of the bar is reduced to 50°, it is observed to elongate afresh, and the movement can readily be followed for over 1,000 hours. If, when the variation has become inappreciable, the temperature of the bar is further reduced to 0°, the elongation once more proceeds.

If the bar is now raised successively to 50° and 100°, being maintained at each of these temperatures in turn for a sufficient time, it recovers sensibly the same lengths as at the termination of the first heatings at these temperatures. The variations in question are thus transitory and slowly reversible.

Again, at ordinary temperatures, elongation may be observed to proceed for a practically indefinite time; this effect is evidently of a different type from those transient and pseudoreversible changes above described. It might at first seem as if it were to be regarded merely as the continuation of those changes the observation of which, in the preceding case, was broken off by the passage to a lower temperature—that, in fact, it is the same effect going on during the permanent subjection to one uniform temperature. But it seems preferable to distinguish between "transient changes" and this last effect, which we may designate a "progressive change." The two types of variation are in all respects analogous to those which have long been recognised as occurring in glass, and which are manifested in the changes of zero of thermometers.

It may be gathered from the brief account just given that the rate at which transient changes occur increases with rise of temperature. I have found that for each rise of 20 degrees the rate of change is multiplied about seven times; which indicates that these changes are to be classed as physicochemical transformations, for which the velocity of reaction increases between two and three times for a rise of temperature of 10 degrees.

The point to be determined—and it is of capital importance in regard to the applications of invar—is the magnitude of these changes. The numbers about to be given will sufficiently indicate this: the elongation on raising the temperature to 100°, after the bar has been recently heated to the forging temperature and cooled in air, is of the order of 30 parts in a million; the complementary elongation observed when the bar is allowed to cool gradually to 0° is of the same order of magnitude. The subsequent progressive change attains, at the expiry of two or three years, a rate of about one part in a million per annum; and after 20 years the rate is about one-third of this. The total progressive change appears to be approximately 20×10^{-6}.

Again, if the bar is cooled or heated in stages, the changes of length may be represented suitably, between 0° and 100°, by a quantity proportional to the square of the temperature reckoned from the ordinary zero. This cannot be regarded as a natural law, but simply as an empirical relation; it enables us to conclude that as the zero temperature is approached the changes become extremely feeble.

On proceeding beyond 100°, the quadratic relation holds for a while; then an inflexion occurs, and the rate of change gradually slows down, to attain its limit at a temperature slightly above 200°.

A knowledge of these phenomena serves to dictate, in the first place, the mode of treatment of samples of invar intended for precise measurements; and, in the second place, to assign the value of the small corrections that have to be appplied to the gross length, as calculated from the instantaneous expansion, in order to allow for progressive or transient changes.

This detail should be added: After a bar of invar has been quenched, the first elongation at 100° is almost double of that which a bar cooled down in air would undergo; but once the bar has been maintained at 100° till it reaches a steady state, all subsequent changes are of the same order, the heating at that temperature having brought the two bars into the same state as regards departure from stability.

The conclusion is that the same correction formulæ are applicable to all bars which have issued from the same casting and been afterwards subjected to different modes of treatment: provided that all alike are subjected to the initial heating for 100 hours at 100°C. Thus if, for example, the task has once been performed of following for a course of years the changes in one bar of a particular melting, the data thus collected will enable one to calculate the extension of any other bar of that melting. Indeed, I possess a curve of the changes observed over a period of 20 years, which applies to a high degree of approximation to all bars of invar of approximately typical composition.

With regard to transient changes it may be added that, according to the quadratic law of temperature, their extent amounts, between 0° and 20°, to 4 per cent. of the total change between 0° and 100°. Adopting for this last the value of 30 parts in a million, there remains outstanding a quantity hardly more than one in a million.

The consideration of these numerical values shows that, though the variations in the case of typical invar may prohibit its use in the construction of standards of reference, it is quite feasible to calculate for every future instant the length of an invar standard, to a degree of approximation adequately ample for the great majority of its applications. If there were need of confirmation one might, at ever increasing intervals of time, compare the invar standard with a reference standard. In the case of a standard not less than 10 years old, a comparison made every three years, for example, would allow of interpolation with certainty, on the express understanding that the bar has undergone an initial stoving (ageing), by being maintained for 100 hours at 100°, and afterwards allowed to cool very gradually over an interval of three months.

As has been stated, the figures given above by way of example refer to invar of typical composition. The phenomena, however, though of the same general character, are governed by coefficients which vary very rapidly with the nickel content of the alloy. As the proportion of nickel approaches 42 per cent., the variations become nil, and then change sign; after passing a negative maximum they again gradually fall off, and cease to be measurable at 70 per cent. of nickel. The coefficient of expansion of the 42 per cent. alloy is about 7×10^{-6}; advantage has been taken of the value of the coefficient—the lowest yet met with in a stable cheap metal—to prepare a considerable number of standards from this alloy.

The study of the ternary alloys has recently led me to the discovery of certain relations, some of which had already been approximately foreseen. Thus the addition of manganese or of chromium lowers the instability for a given content of nickel; there is, however, nothing surprising in this, since the anomaly of expansion diminishes at the same time. It is, however, remarkable that a state of complete stability is reached well before the point at which the anomaly disappears.

The presence of carbon, on the contrary, increases the instability. The effect of variation is such as would make it appear very probable from my experiments, that the instability may be attributed entirely to the presence of carbon. Experiments on this point are, however, lengthy and delicate: they are in full swing, and some months must elapse before it is possible to give a full account of them. But there is little doubt in my mind that they will end in the realisation of an alloy of very low expansibility which will be at the same time absolutely stable.

Outline of Theory.

There is as yet no complete and final theory of the phenomena exhibited by the nickel-steels; it is, however, possible to present a tentative sketch throwing light on the main causes of these phenomena.

Iron and nickel can exist in different states separated by allotropic transformations. In the case of nickel, there is only one transformation known, and that not of large extent. It manifests itself at 360° by a sudden lowering of the magnetic susceptibility, a very slight change of volume, and a bend in the expansion curve, the curvature of which falls to about half its value.

Iron, on the contrary, suffers changes of far greater importance, resulting in the alpha, beta, gamma and delta states. The last of these, discoverd by Curie at a temperature above 1,300°, is of no consequence so far as concerns the properties of the alloys of iron at ordinary temperatures. The beta condition disappears as such in alloys containing a fairly high proportion of a second constituent. Moreover, metallurgists are not in unanimous agreement even as to its separate existence, some considering that it is merely a solution of the other two states in each other.

We may dispense, therefore, with both the beta and the delta states, and speak of the reciprocal passage between the alpha and the gamma states, with this reservation, that in the case of alloys containing small quantities of added metals, this passage occurs with a brief intermediary existence in the beta state.

In the pure metal gamma iron exists unaccompanied at all temperatures above 890°. Up to that point iron expands according to a function affected by an important quadratic term (*AB*, Fig. 11); at that point it undergoes a sudden contraction (*BC*), and then resumes its expansion with a greater coefficient, of value 23×10^{-6}, but with smaller curvature (CD). The linear contraction in the ascending transformation is about 3/1,000. The crystalline form, cubic in the alpha state, becomes octahedral in the gamma state. In this last state the metal is feebly magnetic in the Curie sense (and no longer ferro-magnetic).

Let there be added to the iron a metal with which it forms a solid solution ; according to the principle of cryoscopy, the transformation will be lowered in the scale of temperature, and at the same time it will cease to be sudden, being spread over a considerable interval. The abrupt dip is thus converted into a more gradual curve CE, at each point of which the change of volume accompanying the transformation is superposed on the ordinary expansion. And, the two phenomena being of opposite signs, the result is the negative anomaly revealed by experiment.

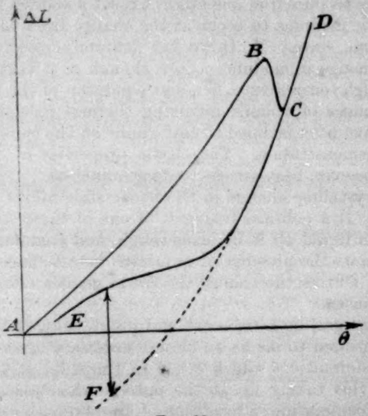

FIG. 11.

Expansion curves of pure iron and of a solid solution.

This theory, applied in detail, gives an adequate explanation of reversible transformation. In the case of irreversible changes another factor intervenes. The intersection of the curves of transformation, discovered by M. Dumas, has been followed out to the fullest extent by M. Chevenard, who has shown that as the nickel content increases, the irreversible transformation diminishes in degree, whilst at the same time the temperature of transformation falls, the transformation vanishing in the alloy of 34·4 per cent. Ni, corresponding to the formula Fe_2Ni. M. Chevenard has thus been led to attribute an important rôle in irreversible transformations to this definite combination which, it may be fortuitously, corresponds very closely to the maximum of the anomaly.

The elastic anomaly may be deduced from that of change of volume by consideration of molecular reactions, which increase considerably as the metal becomes denser. In passing from the alloys to pure iron one might expect a sudden increase in the elastic modulus to occur at the change from the alpha to the gamma state. M. Howe has definitely ascertained that such a change in modulus occurs, though as it takes place at a very high temperature, in a soft condition of the metal, and with changes of volume involving thermal phenomena, one might have been inclined to cast doubt on the convincingness of the demonstration. The elastic properties of the nickel-steels, however, bear witness to its genuineness.

The crystalline change in the irreversible alloys is easy to observe. If a polished fragment of one of these steels is immersed in liquid air it becomes rough, and examination with a lens shows the presence of numerous lancet-shaped crystals of iron. During the change the metal gives a click which is almost musical.

The changes, both transient and progressive, of the nickel-steels appeared to me as an almost necessary consequence of the transformations which occur in the solid solution iron-nickel; this taking up at the outset, when placed at any definite temperature, a condition of approximate equilibrium, changing slowly into one of perfect equilibrium. My recent researches, however, prove that this is not the case. The pure alloys of iron and nickel attain their equilibrium instantaneously; on the contrary, combinations containing carbon—and it is very probable that the same applies to cementite, Fe_3 C, which can exist in two distinct states, separated by a transformation with change of volume—arrive only slowly at their state of equilibrium.

This is, however, beside the main point; a theory relating to the pure nickel-steels is not called upon to include an explanation of this matter.

Applications.

The very remarkable properties of the nickel-steels allow of numerous applications, based either on the anomaly of their expansion, or on that of their elastic properties. In the two domains of measurement of length and of time these applications perhaps call for elucidatory treatment. Only cursory mention need, however, be made of two widely differing applications of these alloys in association with glass, namely, to technical optics and to the manufacture of incandescent lamps.

A region is encountered on the expansibility curve at a nickel content close to 45 per cent. Ni, within which the successive alloys cover the same range of coefficient as the different kinds of glass in general use.* These alloys serve to provide firm mountings for optical apparatus, and leading-in wires for incandescent lamps. In the last case, the wire, if previously freed by long heating *in vacuo* from gas, which is present on considerable quantity, readily welds with the glass. For this purpose it has almost completely replaced platinum, which is thereby set free for other applications. The saving already effected in this way amounts to some millions of pounds.

The Measurement of Length.—The most important of the applications of invar to the measurement of length is the determination of geodesic bases.

During the first three-quarters of the nineteenth century unceasing efforts were made to perfect this measurement regardless of the difficulties and the expense which this entailed. Another tendency then set in, that of simplifying methods, and of making use of longer and more numerous bases. Codifying and systematising diverse processes, M. Edw. Jaderin worked out, about the year 1890, a method consisting in the employment of wires stretched by dynamometers, whereby the respective distances between fiduciary marks mounted on tripods interspersed along the base-line were determined.

But at the time when M. Jaderin first applied his procedure only metals of considerable expansibility were available; and the precise measurement of length consequently involved that of temperature. The uncertainty as to the equality of temperature of a thermometer and of a wire freely exposed to air suggested to him the idea of adapting to his procedure the method of Borda and Lavoisier : namely, the use of a bimetallic standard in the form of two wires, of steel and brass respectively, successively employed to measure the length of each lap or element of length of the base, the difference serving to indicate the temperature.

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* In the descending part of the curve (at a content of about 29 per cent. Ni) this coefficient is the same at ordinary temperature, but owing to the high value of β it rises rapidly as the temperature rises.

On learning of the existence of invar, M. Jaderin, who was at the time preparing to measure bases in connection with the Suedo-Russian Expedition to Spitzberg, requested me to place the necessary wire at his disposal. I was already occupied with this matter, and had caused wire suitable for base line measurements to be manufactured at the steel works at Imphy; this had actually been used in preliminary trials. The effect of wire-drawing on the expansibility was already partially known, and we were consequently able to supply the Mission with wire practically free from temperature-coefficient.

The experiments at Spitzberg, carried out in 1899, proved remarkably successful. A discussion on the question took place at the International Geodesic Conference at Paris in 1900, at which these results, as well as the systematic experiments carried out by M. Benoit and myself at the Bureau International, were discussed. As a consequence a request was made to the International Committee of Weights and Measures that the Bureau should continue its investigations, and undertake the determination of data relating to wires intended for geodesic measurements.

A mural base was immediately erected, and has served for these investigations, which have been continued without a break since 1901.

The number of observations carried out in the course of these 20 years now totals some hundreds of thousands. Our work has been richly rewarded ; it has made it possible to define the conditions under which wires, specially treated, and whose length has been determined at the Bureau or other standardising laboratory, may be wound on drums, transferred to any required spot, unwound, and employed without any modification of their length.

In order to put to fullest advantage the new precision securable by the use of wires of invar, we have designed a complete set of apparatus, made by Carpentier, by means of which one can be certain that in the measurement of a lap, normally of 24 metres, no error exceeding one hundred-thousandth of the length shall be committed. In accordance with the principle of the addition of errors, it follows that a length of 100 laps will never be in error by one part in a million.

Numerous base-line measurements have been conducted in recent years with the aid of invar wire. Checks are obtained, either by making the measurement first one way and then the other, or by successive measurements made with different wires; it has thus been shown that an accuracy of one in a million is quite normal. As the apparatus is easily installed, it is now possible to work over a terrain which would have been quite unsuited to measurements with the older form of apparatus, including, as it did, rigid scales and microscopes. Furthermore, as it is quickly set up, and the laps are longer, and the apparatus is light and readily transported, the staff of observers and assistants can be reduced to one-fifth, and the time required to one-tenth. The total cost is thereby reduced to one-fiftieth of its former amount.

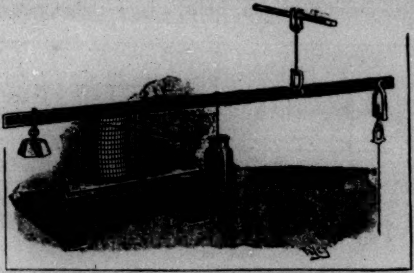

Fig. 12.
Recording apparatus used on the Eiffel Tower.

This economy is reflected in the measurement of angles, since the multiplication of base lines, and their greater length, reduce in considerable proportion the consequences of errors committed in angular measurements; it is thus possible to mitigate the excessive precautions to obtain accuracy which were necessary in the older geodesy.

Vertical Movements of the Eiffel Tower.—When this celebrated tower was erected, the French Service Geographique was concerned with the determination of the amplitude of its oscillatory movement, whether caused by wind or by inequalities of temperature occasioned by solar radiation; but for want of an appropriate method no attempt was then made to study the vertical movements.

By the use of invar wire the problem was solved quite simply. A wire of this material was rigidly attached to the ground at its lower end, and connected at the other to a lever mounted on the second platform of the tower, and which actuated a registering mechanism (Fig. 12). A damper attached to the lever allowed this to return slowly to the position determined by the wire in its straight position, whenever the wire was momentarily curved by a puff of wind. Experiment showed that however violent the air movements, intervals of calm occurred from time to time, of sufficient duration to allow the lever to come to rest. Its limiting positions thus marked the true movements of the tower, whilst a jagged edge, resembling a mop of hair commencing at the envelope of the curve, indicated the effect of the wind.

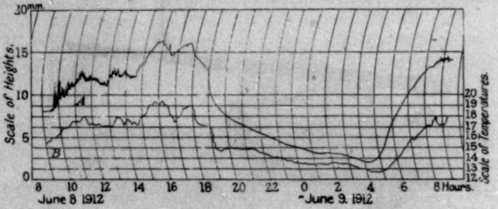

FIG. 13.

The Eiffel Tower. A diurnal record. (A shower of rain at 19·00 has caused a sudden fall in temperature.)

Examples of the kinds of movement that were observed are shown in Figs. 13 and 14. The first of these was obtained with a 24-hour period of revolution of the recording drum; the second with a period of a week. The lower curve in each figure gives the record of temperature.

On inspection of the two series of curves one cannot fail to be struck with the extraordinary correspondence between temperature and displacement, each minute turn in the one finding its counterpart in the other. The Eiffel Tower thus plays the part of a gigantic thermometer, of great sensibility, despite its enormous mass.

The problem solved by invar wire in the case of the tower is a general one. The above application served to show that it was easy to record the movements of a metal structure, whether occasioned by variations of temperature or, as in the case of a bridge carrying traffic, by variations of load.

From the two series of curves the coefficient of expansion of the tower may be calculated.

Let us imagine the tower reduced in scale to a millionth of its actual size. In virtue of its symmetry of structure it would constitute a crystal of the size of a pin's head. Your President would, thanks to the admirable methods which he has devised, and which he has so beautifully applied, be able to determine with ease the coefficient of expansion of a crystal of this minute size. And I must confess, speaking as a metrologist, that I should be far prouder of having determined the expansibility of a pin's head than that of the Eiffel Tower.

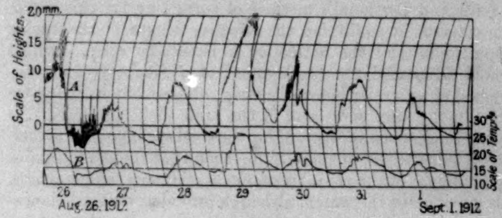

FIG. 14.
The Eiffel Tower. A weekly record. (A stormy week.)

The Measurement of Time.

Clocks.—Of the different methods devised to compensate the pendulum of a clock on account of temperature, by far the most commonly used during the past century was that of Graham, in which the upward expansion of mercury contained in a vessel attached to the rod was made to counteract the downward expansion of the rod itself.

In invar, one finds, according to treatment, all values of the coefficient of expansion between two narrow limits near zero; and having first chosen the material for the rod, it is easy to select that for the bob, provided with screw adjustment, so as to correct by its upward expansion for the elongation of the rod. The advantages of this are manifold. Apart from the presence of a liquid in an oscillating system, Graham's pendulum has the slight defect that it is only truly compensated if the temperature is the same from top to bottom of the clock-case, a condition only sufficiently satisfied in special circumstances.

It may be remarked that for this particular application, the instability of invar is of little consequence, since the going of a clock has in any case to be determined from time to time, in order that any slow variation of its diurnal rate may be accounted for. After the lapse of a few years this variation will have fallen to a few hundredths of a second per annum.

Watches.—A watch provided with a steel spring and a brass balance-wheel loses about 11 seconds per degree rise of temperature per day; and, as we have seen, practically the whole of this variation of rate is due to the change of Young's modulus in the metal composing the spring.

Ferdinand Berthoud's method of correcting for this variation consists in modifying the effective length of the spring automatically by the action of a bimetallic strip; whilst Arnold advocated the employment of a balance-wheel constructed of a circular bimetallic strip, whose moment of inertia alters in the same sense and in the same mean proportion as the modulus of the spring. If for the steel spring one of nickel-steel is substituted, of composition such that the maximum or the minimum of the modulus occurs at ordinary temperatures, it is possible for the watch, although provided with a balance-wheel of one metal only, to possess the same rate at two temperatures situated on either side of the mean of that of the surroundings, say, at 0° and 30°; it would attain a maximum rate, either of losing or gaining, at 15°. The slope of the curve is such that the maximum of the modulus (the minimum point occurs in alloys of low elastic limit) corresponds to a gain of 20 to 25 seconds per day compared with the rate at the extreme temperatures—an amount 12 to 15 times less than that observed when an ordinary steel spring is used. This improvement is fully appreciated by watch manufacturers, as may be judged by the fact that close on three million watches are provided annually with these compensatory springs.

Chronometers.—The balance-wheel of Arnold does not completely compensate the effect of temperature on the rate of a watch. Ferdinand Berthoud in 1775 pointed out a secondary error in his own compensation, and this was rediscovered for Arnold's by Dent in 1832; it consists in the fact that a watch provided with a steel spring, and compensated by a balance-wheel compounded of steel and brass, gains from two to three seconds a day at 15°, assuming the rate of going to be correct at 0° and at 30°.

Considerable efforts have been expended, especially in England, in the endeavour to eliminate the "Dent error." It will be shown that the appropriate use of a nickel-steel allows this result to be simply attained.

The explanation of the Dent error lies in the following fact: The elastic modulus of the spring varies with temperature according to a curve *OH* of decided curvature (Fig. 15A). The action of the balance-wheel, again, is determined by the difference of the expansibilities of brass and steel, *OB* and *OS*, which are expressed by functions in which the coefficients β are very nearly the same. Their difference will, therefore, be represented by a straight line *OC*, and the algebraic sum of *OH* and *OC* will leave a quadratic residue *OD*.

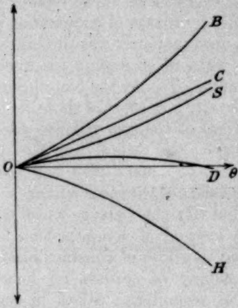

FIG. 15A.
Diagram illustrating the Dent error.

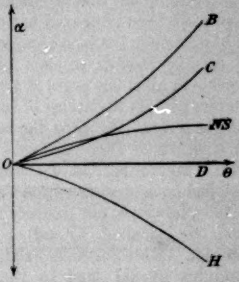

FIG. 15B.
Diagram illustrating the elimination of the Dent error.

Let us now substitute for *OS* a curve *ONS* (Fig. 15B) representing the expansion of a nickel-steel for which β is negative. The difference between it and *OB* is a curve *OC*, which can be made symmetrical with *OH*, so that the algebraic sum *OD* is zero at every point.

In 1899 I worked out calculations relating to the construction of such a balance-wheel, the element of low expansibility being constituted by a steel containing 45 per cent. of nickel; and the same year two Swiss makers of chronometers, M. Nardin and M. Ditisheim, proceeding on the basis of this, succeeded in effecting a compensation which was exact at all temperatures, as was shown by the comparisons made under the direction of Dr. Ad. Hirsch at the observatory of Neuchâtel.

The fact that the expansibility of nickel-steel is sensibly inferior to that of ordinary steel has made it possible to make the strips thicker and shorter, thus reducing decidedly the effects of centrifugal force.

Since the date last mentioned the new balance-wheel has replaced the old one practically everywhere. The records of rate have been improved not only by reason of suppression of the Dent error, but also because, once that error was eliminated, it became feasible to seek for ways of correcting for minor defects in all of the parts of a watch. To-day the best chronometers possess variations of rate only a quarter of those found 20 years ago. At Kew the arrival of the new balance-wheel in a watch of Ditisheim's displaced the record from 92·7 to 94·9. By to-day, at Teddington, the mark 96·9 has been reached by a specimen from the hands of the same maker.

It is, however, not unlikely that this new balance-wheel will soon be equalled, by aid of a method of compensation far simpler. Recent results relating to alloys of constant elastic modulus should lead to the supply of springs no longer in need of correction, primary or secondary. Used in conjunction with balance-wheels of a single metal suitably chosen, they will ensure, in watches provided with them, rates identical at all the ordinary temperatures. It seems hardly conceivable that the mechanism of compensation can be reduced to greater simplicity.

Conclusion.

The results which have been set forth in the course of my lecture lead to a conclusion which can be expressed in the following words: The constant struggle going on between alpha- and gamma-iron in their solid solution with nickel constitutes, in the great diversity of its aspects, one of the happiest phenomena which metallurgy has placed at the disposal of metrologists.

XXXIII. *Convective Cooling and the Theory of Dimensions.*

By Lewis F. Richardson.

(Communicated by C. Chree, LL.D., F.R.S.)

Received February 27, 1920.

The late Lord Rayleigh in a most instructive article on the "Principle of Similitude" ("Nature," Vol. 95, p. 66, 1915) derives a general functional relation for the rate of loss of heat J from a solid of linear dimensions l immersed in viscous fluid moving, at a considerable distance from the solid, with the velocity V. It will be of interest to extend Rayleigh's result so as to include the case in which the velocity near the hot body is partly or entirely produced by the rising of portions of fluid which have been expanded by the heat, as, for instance, in the hot wire ammeter, or as when a kata thermometer is used in calm or nearly calm air.

The essence of the principle of similitude appears to be that phenomena go their way independently of the units whereby we measure them.

A proposed variation in Rayleigh's treatment of the principle was suggested by Riabouchinsky ("Nature," Vol. 95, p. 591, 1915), but was negatived by both Rayleigh and Larmor ("Nature," Vol. 95, p. 644, 1915).

Among the physical variables which affect the phenomenon are the shape of the body and its orientation. These are supposed to be fixed and the others appear to be all comprised in the following list :—

		Dimensions.			
		Length	Mass.	Time.	Temp
Rate of loss of energy by solid	J	2	1	−3	0
Temperature difference between fluid and surface of solid	$(\theta-\theta_a)$	0	0	0	1
Linear dimension of solid	l	1	0	0	0
Velocity of fluid at a distance from solid	V	1	0	−1	0
Thermal capacity of unit mass of fluid ...	γ	2	0	−2	−1
Density of fluid	ρ	−3	1	0	0
Thermal conductivity of fluid	κ	1	1	−3	−1
(accel. of gravity)×(change of density per degree)$=g\partial\rho/\partial\theta=$	β	−2	1	−2	−1
Viscosity of fluid	μ	−1	1	−1	0

Heat energy is supposed to be expressed in work units wherever it occurs explicitly or implicitly.

In this notation Rayleigh's result is

$$J=\kappa l(\theta-\theta_a)F\left\{\frac{lV\gamma\rho}{\kappa},\ \frac{\gamma\mu}{\kappa}\right\},$$

where F is an arbitrary function of the two variables $(lV\gamma\rho)/\kappa$ and $\gamma\mu/\kappa$. To obtain a correspondence with the experimental results of L. V. King * we may put in Rayleigh's formula

$$F=\left\{\mathbf{A}\sqrt{\frac{lV\gamma\rho}{\kappa}}+\mathbf{B}+\mathbf{C}\frac{\gamma\mu}{\kappa}\right\}.$$

Then

$$J=(\theta-\theta_a)l\,\{\mathbf{A}\sqrt{lV\gamma\rho\kappa}+\mathbf{B}\kappa+\mathbf{C}\gamma\mu\}.$$

The terms in A and B are of the form found experimentally by King, but he also found a term $J\propto(\theta-\theta_a)l^2$, which did not involve the velocity. Now it is impossible to choose Rayleigh's F so as to introduce l and to omit V. Thus a wider examination is indicated.

If we form from the nine variables in the list any functions $X_1, X_2 \ldots X_r$ which are of zero dimensions, then any arbitrary relation between $X_1 \ldots X_r$ would satisfy the requirements of the theory of dimensions. And any observed relationship must be expressible as a relation between variables of zero dimensions. Since the variables in the list involve only *powers* of the units of measurements, no generality is lost by making $X_1 \ldots X_r$ severally products of powers of the initial variables. Let U be any such product with indices at first arbitrary, say

$$U=J^{n_1}\,.\,(\theta-\theta_a)^{n_2}\,.\,l^{n_3}\,.\,V^{n_4}\,.\,\gamma^{n_5}\,.\,\rho^{n_6}\,.\,\kappa^{n_7}\,.\,\beta^{n_8}\,.\,\mu^{n_9}. \quad . \quad (1)$$

Then by independence of unit of length

$$0=2n_1 \quad +n_3+n_4+2n_5-3n_6+n_7-2n_8-n_9 \quad . \quad . \quad (2)$$

by mass

$$0=n_1 \quad +n_6+n_7+n_8+n_9 \quad . \quad . \quad (3)$$

by time

$$0=-3n_1 \quad +n_4-2n_5 \quad -3n_7-2n_8-n_9 \quad . \quad (4)$$

by temperature

$$0= \quad n_2 \quad -n_5 \quad -n_7-n_8 \quad . \quad . \quad . \quad (5)$$

These four equations allow us to express four of $n_1 \ldots n_9$ in terms of the other five, which then remain arbitrary. When we come to compare with experiment, it will be convenient to have as arbitrary the indices of those quantities which have

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[*] Phil. Trans. A, Vol. 214 (1914), pp. 373-432. See also equations (9 and (10) of the present Paper.

been varied in the course of the experiment, say, those of J, $(\theta-\theta_a)$, l, V. Solve then equations (2), (3), (4), (5) for n_6, n_7, n_8, n_9. The result is

$$\left.\begin{aligned} n_6 &= +3n_1+4n_2-n_3+2n_4 \\ n_7 &= n_1+3n_2-n_3+n_4-n_5 \\ n_8 &= -n_1-2n_2+n_3-n_4 \\ n_9 &= -4n_1-5n_2+n_3-2n_4+n_5 \end{aligned}\right\} \quad . \; . \; . \; (6)$$

So that U splits up into five factors with arbitrary indices as follows :—

$$U=\left(\frac{J\rho^3\kappa}{\beta\mu^4}\right)^{1}\times\left(\frac{(\theta-\theta_a)\rho^4\kappa^3}{\beta^2\mu^5}\right)^{n_2}\times\left(\frac{l\beta\mu}{\rho\kappa}\right)^{n_3}\times\left(\frac{V\rho^2\kappa}{\beta\mu^2}\right)^{n_4}\times\left(\frac{\gamma\mu}{\kappa}\right)^{n_5}. \quad (7)$$

Each of these factors is, therefore, of zero dimensions. Omitting the indices let us name the factors as follows :—

$$\left.\begin{aligned} &\frac{J\rho^3\kappa}{\beta\mu^4}=X_1\,; \quad \frac{(\theta-\theta_a)\rho^4\kappa^3}{\beta^2\mu^5}=X_2 \\ &\frac{l\beta\mu}{\rho\kappa}=X_3\,; \quad \frac{V\rho^2\kappa}{\beta\mu^2}=X_4\,; \quad \frac{\gamma\mu}{\kappa}=X_5 \end{aligned}\right\} \quad . \; . \; . \; (8)$$

By assigning suitable values to n_1 . . . n_5 we can give to n_6 . . . n_9 any values which are consistent with zero dimensions for U. Therefore, any function of our original nine variables, which is of zero dimensions, can be expressed as a function of X_1 . . . X_5 as defined in (8).

Of recent experimental researches on cooling there is Prof. L. V. King's very thorough study of the loss of heat from fine platinum wires ("Phil. Trans." A., Vol. 214 (1914), pp. 373-432) and Dr. Leonard Hill's calibration of his katathermometer. L. V. King exposed his wires broadside to the current and he expresses his experimental results in a formula, which, if we neglect two very small terms, takes the form

$$J=(\theta-\theta_a)l\{A\sqrt{Vr}+B+Cr\} \quad . \; . \; . \; . \; (9)$$

Here r is the radius of the wire and A, B, C do not depend appreciably on l, r, $(\theta-\theta_a)$ or V. If, to give the wire a constant *shape*, we make l proportional to r, then (9) takes the form

$$J=(\theta-\theta_a)[A'V^{\frac{1}{2}}l^{\frac{3}{2}}+Bl+C'l^2] \quad . \; . \; . \; . \; (10)$$

where A', C' are new constants.

This result must be expressible in terms of X_1 . . . X_5 without other variables.

As J occurs only in X_1, and $(\theta-\theta_a)$ only in X_2, and l only in X_3, and V only in X_4, it follows that equation (10) must be of the form

$$X_1=X_2\{A''X_4^{\frac{1}{2}}\,.\,X_3^{\frac{1}{2}}+B''X_3+C''X^2_3\} \quad . \quad . \quad . \ (11)$$

And as X_5 is the only one of X_1 . . . X_5 which King did not vary experimentally, it follows the A'', B'', C'' may depend on X_5, but, if not, then they are dependent only on the shape of the solid and on its orientation to the flow.

Equation (11) simplifies to

$$J=(\theta-\theta_a)l\left[A''\sqrt{\frac{Vl\rho\kappa^2}{\mu}}+B''\kappa+C''l\frac{\beta\mu}{\rho}\right] \quad . \quad . \quad . \quad . \ (12)$$

King had arrived at the κ in the middle term in the bracket theoretically and had obtained a numerical value agreeing with experiment to 5 per cent. For the first term in the bracket King deduced, from a hydrodynamical theory, the form $\sqrt{Vl\rho\gamma\kappa}$, which we obtain if, remembering that A'' is an unknown function of X_5, we put $A''=X_5^{\frac{1}{2}}A'''$. Thus finally

$$J=(\theta-\theta_a)l\left[A'''\sqrt{Vl\rho\gamma\kappa}+B''\kappa+C''l\,.\,\frac{\beta\mu}{\rho}\right], \quad . \ (13)$$

where A''', B'', C'' may be functions of X_5, of shape and of orientation.

The new result now brought to light is that the last term in the bracket in equation (13) is of the form $\left(l\,\frac{\beta\mu}{\rho}\right)\times$ (a function of X_5). This X_5, be it noted, is much the same for all gases and does not depend on their density over a wide range (*see* Jeans' "Dynamical Theory of Gases," 2nd Ed., p. 317).

Dr. Leonard Hill and his assistants (Medical Research Committee Special Report No. 32) in investigating his "katathermometer" also found that the rate of loss of heat by convection alone varied, at least roughly, as the square root of the pressure.

Hill, Griffith and Flack ("Phil. Trans." Roy. Soc., B.207, p. 197) give a diagram showing a simple proportionality between excess of temperature and rate of loss of heat from convection and radiation jointly; but, as they point out, the linearity is due to the "curvature" of Stefan's radiation law compensating, in this case, the "curvature" due to convection alone. The two curvatures are of opposite sign because these experimenters increased the excess of temperature by cooling the enclosure, instead of by heating the katathermometer. Indeed by measuring Hill's graphs (Figs. 4A and 5, Report *loc. cit.*), I find, quite roughly, that the index σ approaches 1·2.

Although equation (13) holds over a wide range, yet when we remember the complexities of the transition from turbulent to non-turbulent flow, as exhibited by Stanton and Pannell, we must doubt its general validity. *If wider experimental researches give relations between J, $(\theta-\theta_a)$, l, and V, different from those found by King for fine wires, then $X_1 \ldots X_5$ will provide a ready means of generalising the result.*

This inquiry arose at Benson Observatory in connection with the lag of thermometers intended for exploring the upper air. It is communicated by permission of the Director of the Meteorological Office.

I am indebted to my wife for solving equations, to Mr. W. H. Dines for reading the manuscript, and to the Referee of the Physical Society for a number of references.

ABSTRACT.

The Paper consists of an application of the "Principle of Similitude" (Rayleigh, "Nature," Vol. 95, p. 66, 1915) to the loss of heat from hot wires, thermometer bulbs, &c. An equation is obtained which does not involve the dimensions of the bodies and which agrees with the formulæ of King for fine wires and of Hill for thermometer bulbs.

DISCUSSION.

Mr. F. J. W. **Whipple** said he presumed the author took both forced and free convection into account. There were many meteoric problems to which the application of the theory would be of value. Aitken had compared the radiation and convection losses of bodies by exposing them to the sun and measuring their rise in temperature. Using a series of blackened cubes, he had obtained values for the convection which agreed roughly with Barratt's square root law, and would no doubt also fit in with the results of this Paper.

Mr. **Gosling** pointed out that formula No. 13 appeared to make the convection loss increase with the viscosity of the gas. Langmuir had arrived at the opposite conclusion.

Mr. J. **Guild** asked if in applying the theory of dimensions to the heat loss from bodies of different sizes, the considerations were not vitiated by the fact that the size and mean free path of the gas molecules were not varying with the other parts of the apparatus.

The **Author**, in reply, said that the effect mentioned by Mr. Gosling was apparent only. Actually from the nature of the other terms involved, the equation gave reduced values of the convection loss as the viscosity increased. As regards the invariant size of the gas molecules, mean free path, &c., probably the equations cease to hold when any of the dimensions become comparable with the mean free path of the gas.

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